ALGEBRAIC K-THEORY OF RINGS OF INTEGERS IN LOCAL AND GLOBAL FIELDS

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May 18, 2004

Abstract. This survey describes the algebraic K-groups of local and global fields, and the K-groups of rings of integers in these fields. We have used the result of Rost and Voevodsky to determine the odd torsion in these groups.

Introduction

The problem of computing the higher K-theory of a number field F, and of its rings of integers $\mathcal{O}_F$, has a rich history. Since 1972, we have known that the groups $K_n(\mathcal{O}_F)$ are finitely generated [Q3], and known their ranks [Bo], but have only had conjectural knowledge about their torsion subgroups [Li] [Li2] [Bei] until 1997 (starting with [We2]). The resolutions of many of these conjectures by Suslin, Voevodsky, Rost and others have finally made it possible to describe the groups $K_*(\mathcal{O}_F)$. One of the goals of this survey is to give such a description; here is the odd half of the answer (the integers $w_i(F)$ are even, and are defined in section 2):

Theorem 0.1. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F. Then $K_n(\mathcal{O}_S) \cong K_n(F)$ for each odd $n \geq 3$, and these groups are determined only by the number r_1, r_2 of real and complex places of F and the integers $w_i(F)$:*

a) *If F is totally imaginary, $K_n(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$;*

b) *IF F has $r_1 > 0$ real embeddings then, setting $i = (n+1)/2$,*

$$K_n(\mathcal{O}_S) \cong K_n(F) \cong \begin{cases} \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 1 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/2w_i(F) \oplus (\mathbb{Z}/2)^{r_1-1}, & n \equiv 3 \pmod 8 \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/\frac{1}{2}w_i(F), & n \equiv 5 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 7 \pmod 8 \end{cases}$$

In particular, $K_n(\mathbb{Q}) \cong \mathbb{Z}$ for all $n \equiv 5 \pmod 8$ (as $w_i = 2$; see 2.11). More generally, if F has a real embedding and $n \equiv 5 \pmod 8$, then $K_n(F)$ has no 2-primary torsion (because $\frac{1}{2}w_i(F)$ is an odd integer; see 2.8).

The proof of 0.1, will be given in 6.2, 6.5, and section 7 below.

We also know the order of the groups $K_n(\mathbb{Z})$ when $n \equiv 2 \pmod 4$, and know that they are cyclic for $n < 20,000$ (see 7.12 — conjecturally, they are cyclic for every $n \equiv 2$). If B_k denotes the k^{th} Bernoulli number (2.10), and c_k denotes the numerator of $B_k/4k$, then $|K_{4k-2}(\mathbb{Z})|$ is: c_k for k even, and $2c_k$ for k odd; see 7.11.

Although the groups $K_{4k}(\mathbb{Z})$ are conjectured to be zero, at present we only know that these groups have odd order, with no prime factors less than 10^7. This conjecture follows from, and implies, Vandiver's conjecture in number theory (see 8.5 below). In Table 0.2, we have summarized what we know for $n < 20,000$; conjecturally the same pattern holds for all n (see 8.6–8).

$K_0(\mathbb{Z}) = \mathbb{Z}$	$K_8(\mathbb{Z}) = (0?)$	$K_{16}(\mathbb{Z}) = (0?)$	$K_{8a}(\mathbb{Z}) = (0?)$, for $a \geq 1$
$K_1(\mathbb{Z}) = \mathbb{Z}/2$	$K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{8a+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
$K_2(\mathbb{Z}) = \mathbb{Z}/2$	$K_{10}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{18}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{8a+2}(\mathbb{Z}) = \mathbb{Z}/2c_{2a+1}$
$K_3(\mathbb{Z}) = \mathbb{Z}/48$	$K_{11}(\mathbb{Z}) = \mathbb{Z}/1008$	$K_{19}(\mathbb{Z}) = \mathbb{Z}/528$	$K_{8a+3}(\mathbb{Z}) = \mathbb{Z}/2w_{4a+2}$
$K_4(\mathbb{Z}) = 0$	$K_{12}(\mathbb{Z}) = (0?)$	$K_{20}(\mathbb{Z}) = (0?)$	$K_{8a+4}(\mathbb{Z}) = (0?)$
$K_5(\mathbb{Z}) = \mathbb{Z}$	$K_{13}(\mathbb{Z}) = \mathbb{Z}$	$K_{21}(\mathbb{Z}) = \mathbb{Z}$	$K_{8a+5}(\mathbb{Z}) = \mathbb{Z}$
$K_6(\mathbb{Z}) = 0$	$K_{14}(\mathbb{Z}) = 0$	$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$	$K_{8a+6}(\mathbb{Z}) = \mathbb{Z}/c_{2a+2}$
$K_7(\mathbb{Z}) = \mathbb{Z}/240$	$K_{15}(\mathbb{Z}) = \mathbb{Z}/480$	$K_{23}(\mathbb{Z}) = \mathbb{Z}/65520$	$K_{8a+7}(\mathbb{Z}) = \mathbb{Z}/w_{4a+4}$.

Table 0.2. The groups $K_n(\mathbb{Z})$, $n < 20,000$. The notation '(0?)' refers to a finite group, conjecturally zero, whose order is a product of irregular primes $> 10^7$.

For $n \leq 3$, the groups $K_n(\mathbb{Z})$ were known by the early 1970's; see section 1. The right hand sides were also identified as subgroups of $K_n(\mathbb{Z})$ by the late 1970's; see sections 2 and 3. The 2-primary torsion was resolved in 1997 (section 7), but the rest of Table 0.2 only follows from the recent Voevodsky-Rost theorem (sections 6 and 8).

The K-theory of local fields, and global fields of finite characteristic, is richly interconnected with this topic. The other main goal of this article is to survey the state of knowledge here too.

In section 1, we describe the structure of $K_n(\mathcal{O}_F)$ for $n \leq 3$; this material is relatively classical, since these groups have presentations by generators and relations.

The cyclic summands in theorem 0.1 are a special case of a more general construction, due to Harris and Segal. For all fields F, the odd-indexed groups $K_{2i-1}(F)$ have a finite cyclic summand which up to a factor of 2 is detected by a variation of Adams' e-invariant. These summands are discussed in section 2.

There are also canonical free summands related to units, discovered by Borel, and (almost periodic) summands related to the Picard group of R, and the Brauer group of R. These summands were first discovered by Soulé, and are detected by étale Chern classes. These summands are discussed in section 3.

The K-theory of a global field of finite characteristic is handled in section 4. In this case, there is a smooth projective curve X whose higher K-groups are finite, and are related to the action of the Frobenius on the Jacobian variety of X. The orders of these groups are related to the values of the zeta function $\zeta_X(s)$ at negative integers.

The K-theory of a local field E containing $\mathbb{Q}_p$ is handled in section 5. In this case, we understand the p-completion, but do not understand the actual groups $K_*(E)$.

In section 6, we handle the odd torsion in the K-theory of a number field. This is a consequence of the Voevodsky-Rost theorem. These techniques also apply to the 2-primary torsion in totally imaginary number fields, which gives 0.1(a).

The 2-primary torsion in real number fields (those with an embedding in $\mathbb{R}$) is handled in section 7; this material is taken from [RW], and uses Voevodsky's theorem in [V].

Finally, we consider the odd torsion in $K_{2i}(\mathbb{Z})$ in section 8; the odd torsion in $K_{2i-1}(\mathbb{Z})$ is given by 0.1. The torsion occurring in the groups $K_{2i}(\mathbb{Z})$ only involves irregular primes, and is determined by Vandiver's conjecture (8.5). The lack of torsion for regular primes was first guessed by Soulé in [So].

The key technical tool which makes calculations possible for local and global fields is the motivic spectral sequence, from motivic cohomology H^*_M to algebraic K-theory. With coefficients $\mathbb{Z}/m$, the spectral sequence for X is:

$$E_2^{p,q} = H_M^{p-q}(X; \mathbb{Z}/m(-q)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/m). \tag{0.3}$$

This formulation assumes that X is defined over a field [SV]; a similar motivic spectral sequence was established by Levine in [Le, (8.8)] over a Dedekind domain, in which the group $H^n_M(X, \mathbb{Z}(i))$ is defined to be the $(2i-n)$th hypercohomology on X of the complex of higher Chow group sheaves z^i.

When $1/m \in F$, Voevodsky and Rost proved in [V] ($m = 2^\nu$) and [V03] (m odd) that $H^n_M(F, \mathbb{Z}/m(i))$ is isomorphic to $H^n_{\text{ét}}(F, \mu_m^{\otimes i})$ for $n \le i$ and zero if $n > i$. That is, the E_2-terms in this spectral sequence are just étale cohomology groups.

If $X = \text{Spec}(R)$, where R is a Dedekind domain with $F = \text{frac}(R)$ and $1/m \in R$, a comparison of the localization sequences for motivic and étale cohomology (see [Le] and [So, p.268]) shows that $H^n_M(X, \mathbb{Z}/m(i))$ is: $H^n_{\text{ét}}(X, \mu_m^{\otimes i})$ for $n \le i$; the kernel of $H^n_{\text{ét}}(X, \mu_m^{\otimes i}) \to H^n_{\text{ét}}(F, \mu_m^{\otimes i})$ for $n = i+1$; and zero if $n \ge i+2$. That is, the E_2-terms in the fourth quadrant are étale cohomology groups, but there are also modified terms in the column $p = +1$. For example, we have $E_2^{1,-1} = \text{Pic}(X)/m$. This is the only nonzero term in the column $p = +1$ when X has étale cohomological dimension at most two for ℓ-primary sheaves ($cd_\ell(X) \le 2$), as will often occur in this article.

Writing $\mathbb{Z}/\ell^\infty(i)$ for the union of the étale sheaves $\mathbb{Z}/\ell^\infty(i)$, we also obtain a spectral sequence for every field F:

$$E_2^{p,q} = \begin{cases} H^{p-q}_{\text{ét}}(F; \mathbb{Z}/\ell^\infty(-q)) & \text{for } q \le p \le 0, \\ 0 & \text{otherwise} \end{cases} \Rightarrow K_{-p-q}(F; \mathbb{Z}/\ell^\infty), \tag{0.4}$$

and a similar spectral sequence for X which can have nonzero entries in the column $p = +1$. If $cd_\ell(X) \le 2$ it is:

$$E_2^{p,q} = \begin{cases} H^{p-q}_{\text{ét}}(X; \mathbb{Z}/\ell^\infty(-q)) & \text{for } q \le p \le 0, \\ \text{Pic}(X) \otimes \mathbb{Z}/\ell^\infty & \text{for (p,q)=(+1,-1),} \\ 0 & \text{otherwise} \end{cases} \Rightarrow K_{-p-q}(X; \mathbb{Z}/\ell^\infty). \tag{0.5}$$

Periodicity for $\ell = 2$ 0.6. Pick a generator v_1^4 of $\pi^s(S^8; \mathbb{Z}/16) \cong \mathbb{Z}/16$; it defines a generator of $K_8(\mathbb{Z}[1/2]; \mathbb{Z}/16)$ and, by the edge map in (0.3), a canonical element of $H^0_{\text{ét}}(\mathbb{Z}[1/2]; \mu_{16}^{\otimes 4})$ which we shall also call v_1^4. If X is any scheme, smooth over $\mathbb{Z}[1/2]$, the multiplicative pairing of v_1^4 (see [FS] [Le]) with the spectral sequence converging to $K_*(X; \mathbb{Z}/2)$ gives a morphism of spectral sequences $E_r^{p,q} \to E_r^{p-4,q-4}$ from (0.3) to itself. For $p \leq 0$ these maps are isomorphisms, induced by $E_2^{p,q} \cong H^{p-q}_{\text{ét}}(X, \mathbb{Z}/2)$; we shall refer to these isomorphisms as *periodicity isomorphisms*.

Since the Voevodsky-Rost result has not been published yet (see [V03]), it is appropriate for us to indicate exactly where it has been invoked in this survey. In addition to 0.1, 0.2, (0.4) and (0.5) in this introduction, the Voevodsky-Rost theorem is used in theorem 4.7, section 6, 7.10–12 and in section 8.

Acknowledgements

Many people have graciously assisted me in assembling the information in this survey. The author specifically would like to thank Spencer Bloch, Bill Dwyer, Lars Hesselholt, Bruno Kahn, Steve Mitchell, Paul Østvær, John Rognes, Neil Sloane and Christophe Soulé.

The author would also like to thank the Newton Institute, for its hospitality during most of the writing process. The author was also partially supported by an NSA grant.

§1. Classical K-theory of Number Fields

Let F be a number field, *i.e.*, a finite extension of $\mathbb{Q}$, and let $\mathcal{O}_F$ denote the ring of integers in F, *i.e.*, the integral closure of $\mathbb{Z}$ in F. The first few K-groups of F and $\mathcal{O}_F$ have been known since the dawn of K-theory. We quickly review these calculations in this section.

When Grothendieck invented K_0 in the late 1950's, it was already known that over a Dedekind domain R (such as $\mathcal{O}_F$ or the ring $\mathcal{O}_S$ of S-integers in F) every projective module is the sum of ideals, each of which is projective and satisfies $I \oplus J \cong IJ \oplus R$. Therefore $K_0(R) = \mathbb{Z} \oplus \mathrm{Pic}(R)$. Of course, $K_0(F) = \mathbb{Z}$.

In the case $R = \mathcal{O}_F$ the Picard group was already known as the *Class group of* F, and Dirichlet had proven that $\mathrm{Pic}(\mathcal{O}_F)$ is finite. Although not completely understood to this day, computers can calculate the class group for millions of number fields. For cyclotomic fields, we know that $\mathrm{Pic}(\mathbb{Z}[\mu_p]) = 0$ only for $p \le 19$, and that the size of $\mathrm{Pic}(\mathbb{Z}[\mu_p])$ grows exponentially in p; see [Wash].

Example 1.1 (Regular primes). A prime p is called *regular* if $\mathrm{Pic}(\mathbb{Z}[\mu_p])$ has no elements of exponent p, *i.e.*, if p does not divide the order h_p of $\mathrm{Pic}(\mathbb{Z}[\mu_p])$. Kummer proved that this is equivalent to the assertion that p does not divide the numerator of any Bernoulli number B_k, $k \le (p-3)/2$ (see 2.10 and [Wash, 5.34]). Iwasawa proved that a prime p is regular if and only if $\mathrm{Pic}(\mathbb{Z}[\mu_{p^\nu}])$ has no p-torsion for all ν. The smallest irregular primes are $p = 37, 59, 67, 101, 103$ and 131. About 39% of the primes less than 4 million are irregular.

The historical interest in regular primes is that Kummer proved Fermat's Last Theorem for regular primes in 1847. For us, certain calculations of K-groups become easier at regular primes. (See section 8.)

We now turn to units. The valuations on F associated to the prime ideals $\wp$ of $\mathcal{O}_F$ show that the group $F^\times$ is the product of the finite cyclic group $\mu(F)$ of roots of unity and a free abelian group of infinite rank. Dirichlet showed that the group of units of $\mathcal{O}_F$ is the product of $\mu(F)$ and a free abelian group of rank $r_1 + r_2 - 1$, where r_1 and r_2 are the number of embeddings of F into the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$, respectively.

The relation of the units to the class group is given by the "divisor map" (of valuations) from $F^\times$ to the free abelian group on the set of prime ideals $\wp$ in $\mathcal{O}_F$. The divisor map fits into the "Units-Pic" sequence:

$$0 \to \mathcal{O}_F^\times \to F^\times \xrightarrow{\mathrm{div}} \oplus_\wp \mathbb{Z} \to \mathrm{Pic}(\mathcal{O}_F) \to 0.$$

If R is any commutative ring, the group $K_1(R)$ is the product of the group $R^\times$ of units and the group $SK_1(R) = SL(R)/[SL(R), SL(R)]$. Bass-Milnor-Serre proved in [BMS] that $SK_1(R) = 0$ for any ring of S-integers in any global field. Applying this to the number field F we obtain:

$$K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \cong \mu(F) \times \mathbb{Z}^{r_1+r_2-1}. \tag{1.2}$$

For the ring $\mathcal{O}_S$ of S-integers in F, the sequence $1 \to \mathcal{O}_F^\times \to \mathcal{O}_S^\times \to \mathbb{Z}[S] \xrightarrow{\mathrm{div}} \mathrm{Pic}(\mathcal{O}_F) \to \mathrm{Pic}(\mathcal{O}_S) \to 1$ yields:

$$K_1(\mathcal{O}_S) = \mathcal{O}_S^\times \cong \mu(F) \times \mathbb{Z}^{|S|+r_1+r_2-1}. \tag{1.2.1}$$

The 1967 paper [BMS] was instrumental in discovering the group K_2 and its role in number theory. Garland proved in [Gar] that $K_2(\mathcal{O}_F)$ is a finite group. By [Q2], we also know that it is related to $K_2(F)$ by the localization sequence:

$$0 \to K_2(\mathcal{O}_F) \to K_2(F) \xrightarrow{\partial} \oplus_{\wp} k(\wp)^\times \to 0.$$

Since the map ∂ was called the *tame symbol*, the group $K_2(O_F)$ was called the *tame kernel* in the early literature. Matsumoto's theorem allowed Tate to calculate $K_2(\mathcal{O}_F)$ for the quadratic extensions $\mathbb{Q}(\sqrt{-d})$ of discriminant < 35 in [BT]. In particular, we have $K_2(\mathbb{Z}) = K_2\big(\mathbb{Z}\big[\frac{1+\sqrt{-7}}{2}\big]\big) = \mathbb{Z}/2$ on $\{-1,-1\}$, and $K_2(\mathbb{Z}[i]) = 0$.

Tate's key breakthrough, published in [Ta1], was the following result, which was generalized to all fields by Merkurjev and Suslin (in 1982).

Theorem 1.3. *(Tate* [Ta1]*) If F is a number field and R is a ring of S-integers in F such that $1/\ell \in R$ then $K_2(R)/m \cong H^2_{\text{ét}}(R, \mu_m^{\otimes 2})$ for every prime power $m = \ell^\nu$. The ℓ-primary subgroup of $K_2(R)$ is $H^2_{\text{ét}}(R, \mathbb{Z}_\ell(2))$, which equals $H^2_{\text{ét}}(R, \mu_m^{\otimes 2})$ for large ν.*

If F contains a primitive m^{th} root of unity ($m = \ell^\nu$), there is a split exact sequence:

$$0 \to \mathrm{Pic}(R)/m \to K_2(R)/m \to {}_m\mathrm{Br}(R) \to 0.$$

Here ${}_m\mathrm{Br}(R)$ denotes $\{x \in Br(R) | mx = 0\}$. If we compose with the inclusion of $K_2(R)/m$ into $K_2(R; \mathbb{Z}/m)$, Tate's proof shows that the left map $\mathrm{Pic}(R) \to K_2(R; \mathbb{Z}/m)$ is multiplication by the Bott element $\beta \in K_2(R; \mathbb{Z}/m)$ corresponding to a primitive mth root of unity. The quotient ${}_m\mathrm{Br}(R)$ of $K_2(R)$ is easily calculated from the sequence:

$$0 \to \mathrm{Br}(R) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\substack{v \in S \\ \text{finite}}} (\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{1.3.1}$$

Example 1.4. Let $F = \mathbb{Q}(\zeta_{\ell^\nu})$ and $R = \mathbb{Z}[\zeta_{\ell^\nu}, 1/\ell]$, where ℓ is an odd prime and ζ_{ℓ^ν} is a primitive ℓ^νth root of unity. Then R has one finite place, and $r_1 = 0$, so $\mathrm{Br}(R) = 0$ via (1.3.1), and $K_2(R)/\ell \cong \mathrm{Pic}(R)/\ell$. Hence the finite groups $K_2(\mathbb{Z}[\zeta_{\ell^\nu}, 1/\ell])$ and $K_2(\mathbb{Z}[\zeta_{\ell^\nu}])$ have ℓ-torsion if and only if ℓ is an irregular prime.

For the groups $K_n(\mathcal{O}_F)$, $n > 2$, different techniques come into play. Homological techniques were used by Quillen in [Q3] and Borel in [Bo] to prove the following result. Let r_1 (resp., r_2) denote the number of real (resp., complex) embeddings of F; the resulting decomposition of $F \otimes_{\mathbb{Q}} \mathbb{R}$ shows that $[F : \mathbb{Q}] = r_1 + 2r_2$.

Theorem 1.5. *(Quillen-Borel) Let F be a number field. Then the abelian groups $K_n(\mathcal{O}_F)$ are all finitely generated, and their ranks are given by the formula:*

$$\mathit{rank}\ K_n(\mathcal{O}_F) = \begin{cases} r_1 + r_2, & \text{if } n \equiv 1 \pmod 4; \\ r_2, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

In particular, if $n > 0$ is even then $K_n(\mathcal{O}_F)$ is a finite group. If $n = 2i - 1$, the rank of $K_n(\mathcal{O}_F)$ is the order of vanishing of the function ζ_F at $1 - i$.

There is a localization sequence relating the K-theory of $\mathcal{O}_F$, F and the finite fields $\mathcal{O}_F/\wp$; Soulé showed that the maps $K_n(\mathcal{O}_F) \to K_n(F)$ are injections. This proves the following result.

Theorem 1.6. *Let F be a number field.*
a) If $n > 1$ is odd then $K_n(\mathcal{O}_F) \cong K_n(F)$.
b) If $n > 1$ is even then $K_n(\mathcal{O}_F)$ is finite but $K_n(F)$ is an infinite torsion group fitting into the exact sequence

$$0 \to K_n(\mathcal{O}_F) \to K_n(F) \to \bigoplus_{\wp \subset \mathcal{O}_F} K_{n-1}(\mathcal{O}_F/\wp) \to 0.$$

For example, the groups $K_3(\mathcal{O}_F)$ and $K_3(F)$ are isomorphic, and hence the direct sum of $\mathbb{Z}^{r_2}$ and a finite group. The Milnor K-group $K_3^M(F)$ is isomorphic to $(\mathbb{Z}/2)^{r_1}$ by [BT], and injects into $K_3(F)$ by [MS].

The following theorem was proven by Merkurjev and Suslin in [MS]. Recall that F is said to be *totally imaginary* if it cannot be embedded into $\mathbb{R}$, *i.e.*, if $r_1 = 0$ and $r_2 = [F : \mathbb{Q}]/2$. The positive integer $w_2(F)$ is defined in section 2 below, and is always divisible by 24.

Theorem 1.7. *(Structure of K_3F) Let F be a number field, and set $w = w_2(F)$.*
a) If F is totally imaginary, then $K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$;
b) If F has a real embedding then $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$ is a subgroup of $K_3(F)$ and:

$$K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/(2w) \oplus (\mathbb{Z}/2)^{r_1 - 1}.$$

Examples 1.7.1. a) When $F = \mathbb{Q}$ we have $K_3(\mathbb{Z}) = K_3(\mathbb{Q}) \cong \mathbb{Z}/48$, because $w_2(F) = 24$. This group was first calculated by Lee and Szcarba.

b) When $F = \mathbb{Q}(i)$ we have $w_2(F) = 24$ and $K_3(\mathbb{Q}(i)) \cong \mathbb{Z} \oplus \mathbb{Z}/24$.

c) When $F = \mathbb{Q}(\sqrt{\pm 2})$ we have $w_2(F) = 48$ because $F(i) = \mathbb{Q}(\zeta_8)$. For these two fields, $K_3(\mathbb{Q}(\sqrt{2}) \cong \mathbb{Z}/96 \oplus \mathbb{Z}/2$, while $K_3(\mathbb{Q}(\sqrt{-2}) \cong \mathbb{Z} \oplus \mathbb{Z}/48$.

Classical techniques have not been able to proceed much beyond this. Although Bass and Tate showed that the Milnor K-groups $K_n^M(F)$ are $(\mathbb{Z}/2)^{r_1}$ for all $n \geq 3$, and hence nonzero for every real number field (one embeddable in $\mathbb{R}$, so that $r_1 \neq 0$), we have the following discouraging result.

Lemma 1.8. *Let F be a real number field. The map $K_4^M(F) \to K_4(F)$ is not injective, and the map $K_n^M(F) \to K_n(F)$ is zero for $n \geq 5$.*

Proof. The map $\pi_1^s \to K_1(\mathbb{Z})$ sends η to $[-1]$. Since $\pi_*^s \to K_*(\mathbb{Z})$ is a ring homomorphism and $\eta^4 = 0$, the Steinberg symbol $\{-1,-1,-1,-1\}$ must be zero in $K_4(\mathbb{Z})$. But the corresponding Milnor symbol is nonzero in $K_4^M(F)$, because it is nonzero in $K_4^M(\mathbb{R})$. This proves the first assertion. Bass and Tate prove [BT] that $K_n^M(F)$ is in the ideal generated by $\{-1,-1,-1,-1\}$ for all $n \geq 5$, which gives the second assertion. □

Remark 1.9. Around the turn of the century, homological calculations by Rognes [R4] and Elbaz-Vincent/Gangl/Soulé [EGS] proved that $K_4(\mathbb{Z}) = 0$, $K_5(\mathbb{Z}) = \mathbb{Z}$, and that $K_6(\mathbb{Z})$ has at most 3-torsion. These follow from a refinement of the calculations by Lee-Szczarba and Soulé in [So1] that there is no p-torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$ for $p > 3$, together with the calculation in [RW] that there is no 2-torsion in $K_4(\mathbb{Z})$, $K_5(\mathbb{Z})$ or $K_6(\mathbb{Z})$.

The results of Rost and Voevodsky imply that $K_7(\mathbb{Z}) \cong \mathbb{Z}/240$ (see [We2]). It is still an open question whether or not $K_8(\mathbb{Z}) = 0$.

§2. The e-invariant

The odd-indexed K-groups of a field F have a canonical torsion summand, discovered by Harris and Segal in [HS]. It is detected by a map called the e-invariant, which we now define.

Let F be a field, with separable closure $\bar{F}$ and Galois group $G = \mathrm{Gal}(\bar{F}/F)$. The abelian group μ of all roots of unity in $\bar{F}$ is a G-module. For all i, we shall write $\mu(i)$ for the abelian group μ, made into a G-module by letting $g \in G$ act as $\zeta \mapsto g^i(\zeta)$. (This modified G-module structure is called the ith Tate twist of the usual structure.) Note that the abelian group underlying $\mu(i)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ if $\mathrm{char}(F) = 0$ and $\mathbb{Q}/\mathbb{Z}[1/p]$ if $\mathrm{char}(F) = p \neq 0$. For each prime $\ell \neq \mathrm{char}(F)$, we write $\mathbb{Z}/\ell^\infty(i)$ for the ℓ-primary G-submodule of $\mu(i)$, so that $\mu(i) = \oplus \mathbb{Z}/\ell^\infty(i)$.

For each odd $n = 2i-1$, Suslin proved [Su1, Su2] that the torsion subgroup of $K_{2i-1}\bar{F}$ is naturally isomorphic to $\mu(i)$. It follows that there is a natural map

$$e : K_{2i-1}(F)_{\mathrm{tors}} \to K_{2i-1}(\bar{F})^G_{\mathrm{tors}} \cong \mu(i)^G. \tag{2.1}$$

If $\mu(i)^G$ is a finite group, we write $w_i(F)$ for its order, so that $\mu(i)^G \cong \mathbb{Z}/w_i(F)$. This is the case for all local and global fields (by 2.3.1 below). We shall call e the *e-invariant*, since the composition $\pi^s_{2i-1} \to K_{2i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_i(\mathbb{Q})$ is Adams' complex e-invariant by [Q5].

The target group $\mu(i)^G$ is always the direct sum of its ℓ-primary Sylow subgroups $\mathbb{Z}/\ell^\infty(i)^G$. The orders of these subgroups are determined by the roots of unity in the cyclotomic extensions $F(\mu_{\ell^\nu})$. Here is the relevant definition.

Definition 2.2. Fix a prime ℓ. For any field F, we define integers $w_i^{(\ell)}(F)$ by

$$w_i^{(\ell)}(F) = \max\{\ell^\nu \mid \mathrm{Gal}(F(\mu_{\ell^\nu})/F) \text{ has exponent dividing } i\}$$

for each integer i. If there is no maximum ν we set $w_i^{(\ell)}(F) = \ell^\infty$.

Lemma 2.3. *Let F be a field and set $G = \mathrm{Gal}(\bar{F}/F)$. Then $\mathbb{Z}/\ell^\infty(i)^G$ is isomorphic to $\mathbb{Z}/w_i^{(\ell)}(F)$. Thus the target of the e-invariant is $\bigoplus_\ell \mathbb{Z}/w_i^{(\ell)}(F)$.*

Suppose in addition that $w_i^{(\ell)}(F)$ is 1 for almost all ℓ, and is finite otherwise. Then the target of the e-invariant is $\mathbb{Z}/w_i(F)$, where $w_i(F) = \prod w_i^{(\ell)}(F)$.

Proof. Let ζ be a primitive ℓ^νth root of unity. Then $\zeta^{\otimes i}$ is invariant under $g \in G$ (the absolute Galois group) precisely when $g^i(\zeta) = \zeta$, and $\zeta^{\otimes i}$ is invariant under all of G precisely when the group $\mathrm{Gal}(F(\mu_{\ell^\nu})/F)$ has exponent i. □

Corollary 2.3.1. *Suppose that $F(\mu_\ell)$ has only finitely many ℓ-primary roots of unity for all primes ℓ, and that $[F(\mu_\ell) : F] \to \infty$ as $\ell \to \infty$. Then the $w_i(F)$ are finite for all $i \neq 0$.*

This is the case for all local and global fields.

Proof. For fixed $i \neq 0$, the formulas in 2.7 and 2.8 below show that each $w_i^{(\ell)}$ is finite, and equals one except when $[F(\mu_\ell) : F]$ divides i. By assumption, this exception happens for only finitely many ℓ. Hence $w_i(F)$ is finite. □

Example 2.4 (finite fields). Consider a finite field $\mathbb{F}_q$. It is a pleasant exercise to show that $w_i(\mathbb{F}_q) = q^i - 1$ for all i. Quillen computed the K-theory of $\mathbb{F}_q$ in [Q1], showing that $K_{2i}(\mathbb{F}_q) = 0$ for $i > 0$ and that $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/w_i(\mathbb{F}_q)$. In this case, the e-invariant is an isomorphism.

The key part of the following theorem, *i.e.*, the existence of a $\mathbb{Z}/w_i$ summand, was discovered in the 1975 paper [HS] by Harris and Segal; the splitting map was constructed in an ad hoc manner for number fields (see 2.5.2 below). The canonical nature of the splitting map was only established much later, in [DFM], [Ham] and [Ka2].

The summand does not always exist when $\ell = 2$; for example $K_5(\mathbb{Z}) = \mathbb{Z}$ but $w_3(\mathbb{Q}) = 2$. The Harris-Segal construction fails when the Galois groups of cyclotomic field extensions are not cyclic. With this in mind, we call a field F *non–exceptional* if the Galois groups $\mathrm{Gal}(F(\mu_{2^\nu})/F)$ are cyclic for every ν, and *exceptional* otherwise. There are no exceptional fields of finite characteristic. Both $\mathbb{R}$ and $\mathbb{Q}_2$ are exceptional, and so are each of their subfields. In particular, real number fields (like $\mathbb{Q}$) are exceptional, and so are some totally imaginary number fields, like $\mathbb{Q}(\sqrt{-7})$.

Theorem 2.5. *Let R be an integrally closed domain containing $1/\ell$, and set $w_i = w_i^{(\ell)}(R)$. If $\ell = 2$, we suppose that R is non-exceptional. Then each $K_{2i-1}(R)$ has a canonical direct summand isomorphic to $\mathbb{Z}/w_i$, detected by the e-invariant.*

The splitting $\mathbb{Z}/w_i \to K_{2i-1}(R)$ is called the Harris-Segal map, and its image is called the Harris-Segal summand of $K_{2i-1}(R)$.

Example 2.5.1. If R contains a primitive ℓ^νth root of unity ζ, we can give a simple description of the subgroup $\mathbb{Z}/\ell^\nu$ of the Harris-Segal summand. In this case, $H^0_{\text{ét}}(R, \mu_{\ell^\nu}^{\otimes i}) \cong \mu_{\ell^\nu}^{\otimes i}$ is isomorphic to $\mathbb{Z}/\ell^\nu$, on generator $\zeta \otimes \cdots \otimes \zeta$. If $\beta \in K_2(R; \mathbb{Z}/\ell^\nu)$ is the *Bott element* corresponding to ζ, the *Bott map* $\mathbb{Z}/\ell^\nu \to K_{2i}(R; \mathbb{Z}/\ell^\nu)$ sends 1 to β^i. (This multiplication is defined unless $\ell^\nu = 2$[1].) The Harris-Segal map, restricted to $\mathbb{Z}/\ell^\nu \subseteq \mathbb{Z}/m$, is just the composition

$$\mu_{\ell^\nu}^{\otimes i} \cong \mathbb{Z}/\ell^\nu \xrightarrow{\text{Bott}} K_{2i}(R; \mathbb{Z}/\ell^\nu) \to K_{2i-1}(R).$$

Remark 2.5.2. Harris and Segal [HS] originally constructed the Harris-Segal map by studying the homotopy groups of the space BN^+, where N is the union of the wreath products $\mu \wr \Sigma_n$, $\mu = \mu_{\ell^\nu}$. Each wreath product embeds in $GL_n(R[\zeta_{\ell^\nu}])$ as the group of matrices whose entries are either zero or ℓ^νth roots of unity, each row and column having at most one nonzero entry. Composing with the transfer, this gives a group map $N \to GL(R[\zeta_{\ell^\nu}]) \to GL(R)$ and hence a topological map $BN^+ \to GL(R)^+$.

From a topological point of view, BN^+ is the zeroth space of the spectrum $\Sigma^\infty(B\mu_+)$, and is also the K-theory space of the symmetric monoidal category of finite free μ-sets. The map of spectra underlying $BN^+ \to GL(R)^+$ is obtained by taking the K-theory of the free R-module functor from finite free μ-sets to free R-modules.

Harris and Segal split this map by choosing a prime p which is primitive mod ℓ, and is a topological generator of $\mathbb{Z}_\ell^\times$. Their argument may be interpreted as saying that if $\mathbb{F}_q = \mathbb{F}_p[\zeta_{\ell^\nu}]$ then the composite map $\Sigma^\infty(B\mu_+) \to K(R) \to K(\mathbb{F}_q)$ is an equivalence after KU-localization.

If F is an exceptional field, a transfer argument using $F(\sqrt{-1})$ shows that there is a cyclic summand in $K_{2i-1}(R)$ whose order is either $w_i(F)$, $2w_i(F)$ or $w_i(F)/2$. If F is a totally imaginary number field, we will see in 6.5 that the Harris-Segal summand is always $\mathbb{Z}/w_i(F)$. The following theorem, proven in section 7.5 below (see [RW]), shows that all possibilities occur for real number fields, *i.e.*, number fields embeddable in $\mathbb{R}$.

Theorem 2.6. *Let F be a real number field. Then the Harris-Segal summand in $K_{2i-1}(\mathcal{O}_F)$ is isomorphic to:*

(1) $\mathbb{Z}/w_i(F)$, *if* $i \equiv 0 \pmod 4$ *or* $i \equiv 1 \pmod 4$, *i.e.,* $2i-1 \equiv \pm 1 \pmod 8$;
(2) $\mathbb{Z}/2w_i(F)$, *if* $i \equiv 2 \pmod 4$, *i.e.,* $2i-1 \equiv 3 \pmod 8$;
(3) $\mathbb{Z}/\frac{1}{2}w_i(F)$, *if* $i \equiv 3 \pmod 4$, *i.e.,* $2i-1 \equiv 5 \pmod 8$.

Here are the formulas for the numbers $w_i^{(\ell)}(F)$, taken from [HS, p. 28], and from [W1, 6.3] when $\ell = 2$. Let $\log_\ell(n)$ be the maximal power of ℓ dividing n, *i.e.*, the ℓ-adic valuation of n. By convention let $\log_\ell(0) = \infty$.

Proposition 2.7. *Fix a prime $\ell \neq 2$, and let F be a field of characteristic $\neq \ell$. Let a be maximal such that $F(\mu_\ell)$ contains a primitive ℓ^ath root of unity. Then if $r = [F(\mu_\ell) : F]$ and $b = \log_\ell(i)$ the numbers $w_i^{(\ell)} = w_i^{(\ell)}(F)$ are:*
(a) If $\mu_\ell \in F$ then $w_i^{(\ell)} = \ell^{a+b}$;
(b) If $\mu_\ell \notin F$ and $i \equiv 0 \pmod r$ then $w_i^{(\ell)} = \ell^{a+b}$;
(c) If $\mu_\ell \notin F$ and $i \not\equiv 0 \pmod r$ then $w_i^{(\ell)} = 1$.

Proof. Since ℓ is odd, $G = \mathrm{Gal}(F(\mu_{\ell^{a+\nu}})/F)$ is a cyclic group of order $r\ell^\nu$ for all $\nu \geq 0$. If a generator of G acts on $\mu_{\ell^{a+\nu}}$ by $\zeta \mapsto \zeta^g$ for some $g \in (\mathbb{Z}/\ell^{a+\nu})^\times$ then it acts on $\mu^{\otimes i}$ by $\zeta \mapsto \zeta^{g^i}$. $\square$

Example 2.7.1. If $F = \mathbb{Q}(\mu_{p^\nu})$ and $\ell \neq 2, p$ then $w_i^{(\ell)}(F) = w_i^{(\ell)}(\mathbb{Q})$ for all i. This number is 1 unless $(\ell-1)|i$; if $(\ell-1)|i$ but $\ell \not| i$ then $w_i^{(\ell)}(F) = \ell$. In particular, if $\ell = 3$ and $p \neq 3$ then $w_i^{(3)}(F) = 1$ for odd i, and $w_i^{(3)}(F) = 3$ exactly when $i \equiv 2, 4 \pmod 6$. Of course, $p|w_i(F)$ for all i.

Proposition 2.8. *($\ell = 2$) Let F be a field of characteristic $\neq 2$. Let a be maximal such that $F(\sqrt{-1})$ contains a primitive 2^ath root of unity. Let i be any integer, and let $b = \log_2(i)$. Then the 2-primary numbers $w_i^{(2)} = w_i^{(2)}(F)$ are:*
(a) If $\sqrt{-1} \in F$ then $w_i^{(2)} = 2^{a+b}$ for all i.
(b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)} = 2$.
(c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)} = 2^{a+b}$.
(d) If $\sqrt{-1} \notin F$, F is non–exceptional and i is even then $w_i^{(2)} = 2^{a+b-1}$.

Example 2.9 (local fields). If E is a local field, finite over $\mathbb{Q}_p$, then $w_i(E)$ is finite by 2.3.1. Suppose that the residue field is $\mathbb{F}_q$. Since (for $\ell \neq p$) the number of ℓ-primary roots of unity in $E(\mu_\ell)$ is the same as in $\mathbb{F}_q(\mu_\ell)$, we see from 2.7 and 2.8 that $w_i(E)$ is $w_i(\mathbb{F}_q) = q^i - 1$ times a power of p.

If $p > 2$ the p-adic rational numbers $\mathbb{Q}_p$ have $w_i(\mathbb{Q}_p) = q^i - 1$ unless $(p-1)|i$; if $i = (p-1)p^b m$ $(p \nmid m)$ then $w_i(\mathbb{Q}_p) = (q^i - 1)p^{1+b}$.

For $p = 2$ we have $w_i(\mathbb{Q}_2) = 2(2^i - 1)$ for i odd, because $\mathbb{Q}_2$ is exceptional, and $w_i(\mathbb{Q}_2) = (2^i - 1)2^{2+b}$ for i even, $i = 2^b m$ with m odd.

Bernoulli numbers 2.10. The numbers $w_i(\mathbb{Q})$ are related to the *Bernoulli numbers* B_k. These were defined by Jacob Bernoulli in 1713 as coefficients in the power series

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{t^{2k}}{(2k)!}.$$

(We use the topologists' B_k from [MSt], all of which are positive. Number theorists would write it as $(-1)^{k+1}B_{2k}$.) The first few Bernoulli numbers are:

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, B_6 = \frac{691}{2730}, B_7 = \frac{7}{6}, B_8 = \frac{3617}{510}.$$

The denominator of B_k is alway squarefree, divisible by 6, and equal to the product of all primes with $(p-1)|2k$. Moreover, if $(p-1) \nmid 2k$ then p is not in the denominator of B_k/k even if $p|k$; see [MSt].

Although the numerator of B_k is difficult to describe, Kummer's congruences show that if p is regular it does not divide the numerator of *any* B_k/k (see [Wash, 5.14]). Thus only irregular primes can divide the numerator of B_k/k (see 1.1).

Remark 2.10.1. We have already remarked in 1.1 that if a prime p divides the numerator of some B_k/k then p divides the order of $\mathrm{Pic}(\mathbb{Z}[\mu_p])$. Bernoulli numbers also arise as values of the Riemann zeta function. Euler proved (in 1735) that $\zeta_{\mathbb{Q}}(2k) = B_k(2\pi)^{2k}/2(2k)!$. By the functional equation, we have $\zeta_{\mathbb{Q}}(1-2k) = (-1)^k B_k/2k$. Thus the denominator of $\zeta(1-2k)$ is $\frac{1}{2}w_{2k}(\mathbb{Q})$.

Remark 2.10.2. The Bernoulli numbers are of interest to topologists because if $n = 4k-1$ the image of $J : \pi_n SO \to \pi_n^s$ is cyclic of order equal to the denominator of $B_k/4k$, and the numerator determines the number of exotic $(4k-1)$-spheres which bound parallizable manifolds; see [MSt, App.B].

From 2.10, 2.7 and 2.8 it is easy to verify the following important result.

Lemma 2.11. *If i is odd then $w_i(\mathbb{Q}) = 2$ and $w_i(\mathbb{Q}(\sqrt{-1})) = 4$. If $i = 2k$ is even then $w_i(\mathbb{Q}) = w_i(\mathbb{Q}(\sqrt{-1}))$, and this integer is the denominator of $B_k/4k$. The prime ℓ divides $w_i(\mathbb{Q})$ exactly when $(\ell - 1)$ divides i.*

Example 2.11.1. For $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$, $w_2 = 24$, $w_4 = 240$, $w_6 = 504 = 2^3 \cdot 3^2 \cdot 7$, $w_8 = 480 = 2^5 \cdot 3 \cdot 5$, $w_{10} = 264 = 2^3 \cdot 3 \cdot 11$, and $w_{12} = 65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$.

The w_i are the orders of the Harris-Segal summands of $K_3(\mathbb{Q}[\sqrt{-1}])$, $K_7(\mathbb{Q}[\sqrt{-1}])$, ..., $K_{23}(\mathbb{Q}[\sqrt{-1}])$ by 2.5. In fact, we will see in 6.5 that $K_{2i-1}(\mathbb{Q}[\sqrt{-1}]) \cong \mathbb{Z} \oplus \mathbb{Z}/w_i$ for all $i \geq 2$.

By 2.6, the orders of the Harris-Segal summands of $K_7(\mathbb{Q})$, $K_{15}(\mathbb{Q})$, $K_{23}(\mathbb{Q})$, ... are w_4, w_8, w_{12}, etc., and the orders of the Harris-Segal summands of $K_3(\mathbb{Q})$, $K_{11}(\mathbb{Q})$, $K_{19}(\mathbb{Q})$, ... are $2w_2 = 48$, $2w_6 = 1008$, $2w_{10} = 2640$, etc. In fact, these summands are exactly the torsion subgroups of the groups $K_{2i-1}(\mathbb{Q})$.

Example 2.12. The image of the natural maps $\pi_n^s \to K_n(\mathbb{Z})$ capture most of the Harris-Segal summands, and were analyzed by Quillen in [Q5]. When n is $8k+1$ or $8k+2$, there is a $\mathbb{Z}/2$-summand in $K_n(\mathbb{Z})$, generated by the image of Adams' element μ_n. (It is the 2-torsion subgroup by [We2].) Since $w_{4k+1}(\mathbb{Q}) = 2$, we may view it as the Harris-Segal summand when $n = 8k+1$. When $n = 8k+5$, the Harris-Segal summand is zero by 2.6. When $n = 8k+7$ the Harris-Segal summand of $K_n(\mathbb{Z})$ is isomorphic to the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+4}(\mathbb{Q})$ of π_n^s.

When $n = 8k+3$, the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of π_n^s is contained in the Harris-Segal summand $\mathbb{Z}/(2w_i)$ of $K_n(\mathbb{Z})$; the injectivity was proven by Quillen in [Q5], and Browder showed that the order of the summand was $2w_i(\mathbb{Q})$.

Not all of the image of J injects into $K_*(\mathbb{Z})$. If $n = 0, 1 \pmod 8$ then $J(\pi_n O) \cong \mathbb{Z}/2$, but Waldhausen showed (in 1982) that these elements map to zero in $K_n(\mathbb{Z})$.

Example 2.13. Let $F = \mathbb{Q}(\zeta + \zeta^{-1})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta)$, $\zeta^p = 1$ with p odd. Then $w_i(F) = 2$ for odd i, and $w_i(F) = w_i(\mathbb{Q}(\zeta))$ for even $i > 0$ by 2.7 and 2.8. Note that $p | w_i(F[\zeta])$ for all i, $p | w_i(F)$ if and only if i is even, and $p | w_i(\mathbb{Q})$ only when $(p-1) | i$. If $n \equiv 3 \pmod 4$, the groups $K_n(\mathbb{Z}[\zeta+\zeta^{-1}]) = K_n(F)$ are finite by 1.5; the order of their Harris-Segal summands are given by theorem 2.6, and have an extra p-primary factor not detected by the image of J when $n \not\equiv -1 \pmod{2p-2}$.

Birch-Tate Conjecture 2.14. If F is a number field, the zeta function $\zeta_F(s)$ has a pole of order r_2 at $s = -1$. Birch and Tate [Ta] conjectured that for totally real number fields ($r_2 = 0$) we have

$$\zeta_F(-1) = (-1)^{r_1} |K_2(\mathcal{O}_F)| / w_2(F).$$

The odd part of this conjecture was proven by Wiles in [Wi], using Tate's theorem 1.3. The two-primary part is still open, but it is known to be a consequence of the 2-adic Main Conjecture of Iwasawa Theory (see Kolster's appendix to [RW]), which was proven by Wiles in *loc. cit.* for abelian extensions of $\mathbb{Q}$. Thus the full Birch-Tate Conjecture holds for all abelian extensions of $\mathbb{Q}$. For example, when $F = \mathbb{Q}$ we have $\zeta_{\mathbb{Q}}(-1) = -1/12$, $|K_2(\mathbb{Z})| = 2$ and $w_2(\mathbb{Q}) = 24$.

§3. Étale Chern classes

We have seen in 1.2 and 1.3 that $H^1_{\text{ét}}$ and $H^2_{\text{ét}}$ are related to K_1 and K_2. In order to relate them to higher K-theory, it is useful to have well-behaved maps. In one direction, we use the étale Chern classes introduced in [So], but in the form found in Dwyer-Friedlander [DF].

In this section, we construct the maps in the other direction. Our formulation is due to Kahn [Ka, Ka1, Ka2]; they were introduced in [Ka1], where they were called "anti-Chern classes." Kahn's maps are an efficient reorganization of the constructions of Soulé [So] and Dwyer-Friedlander [DF]. Of course, there are higher Kahn maps, but we do not need them for local or global fields so we omit them here.

If F is a field containing $1/\ell$, there is a canonical map from $K_{2i-1}(F;\mathbb{Z}/\ell^\nu)$ to $H^1_{\text{ét}}(F,\mu_{\ell^\nu}^{\otimes i})$, called the *first étale Chern class*. It is the composition of the map to the étale K-group $K^{\text{ét}}_{2i-1}(F;\mathbb{Z}/\ell^\nu)$ followed by the edge map in the Atiyah-Hirzebruch spectral sequence for étale K-theory [DF]. For $i=1$ it is the Kummer isomorphism from $K_1(F;\mathbb{Z}/\ell^\nu) = F^\times/F^{\times\ell^\nu}$ to $H^1_{\text{ét}}(F,\mu_{\ell^\nu})$.

For each i and ν, we can construct a splitting of the first étale Chern class, at least if ℓ is odd (or $\ell=2$ and F is non-exceptional). Let F_ν denote the smallest field extension of F over which the Galois module $\mu_{\ell^\nu}^{\otimes i-1}$ is trivial, and let Γ_ν denote the Galois group of F_ν over F. Kahn proved in [Ka] that the transfer map induces an isomorphism $H^1_{\text{ét}}(F_\nu,\mu_{\ell^\nu}^{\otimes i})_{\Gamma_\nu} \xrightarrow{\cong} H^1_{\text{ét}}(F,\mu_{\ell^\nu}^{\otimes i})$. Note that because $H^1_{\text{ét}}(F_\nu,\mu_{\ell^\nu}) \cong F_\nu^\times/\ell^\nu$ we have an isomorphism of Γ_ν-modules $H^1_{\text{ét}}(F_\nu,\mu_{\ell^\nu}^{\otimes i}) \cong (F_\nu^\times)\otimes\mu_{\ell^\nu}^{\otimes i-1}$.

Definition 3.1. The *Kahn map* $H^1_{\text{ét}}(F,\mu_{\ell^\nu}^{\otimes i}) \to K_{2i-1}(F;\mathbb{Z}/\ell^\nu)$ is the composition

$$H^1_{\text{ét}}(F,\mu_{\ell^\nu}^{\otimes i}) \xleftarrow{\cong} H^1_{\text{ét}}(F_\nu,\mu_{\ell^\nu}^{\otimes i})_{\Gamma_\nu} = \left[F_\nu^\times\otimes\mu_{\ell^\nu}^{\otimes i-1}\right]_{\Gamma_\nu} \xrightarrow{\text{Harris-Segal}}$$

$$\to \left[(F_\nu^\times)\otimes K_{2i-2}(F_\nu;\mathbb{Z}/\ell^\nu)\right]_{\Gamma_\nu} \xrightarrow{\cup} K_{2i-1}(F_\nu;\mathbb{Z}/\ell^\nu)_{\Gamma_\nu} \xrightarrow{\text{transfer}} K_{2i-1}(F;\mathbb{Z}/\ell^\nu).$$

Compatibility 3.1.1. Let F be the quotient field of a discrete valuation ring whose residue field k contains $1/\ell$. Then the Kahn map is compatible with the Harris-Segal map in the sense that for $m=\ell^\nu$ the diagram commutes.

$$\begin{array}{ccc} H^1_{\text{ét}}(F,\mu_m^{\otimes i}) & \xrightarrow{\partial} & H^0_{\text{ét}}(k,\mu_m^{\otimes i-1}) \\ \downarrow{\scriptstyle \text{Kahn}} & & \downarrow{\scriptstyle \text{Harris-Segal}} \\ K_{2i-1}(F;\mathbb{Z}/m) & \xrightarrow{\partial} & K_{2i-2}(k;\mathbb{Z}/m) \end{array}$$

To see this, one immediately reduces to the case $F=F_\nu$. In this case, the Kahn map is the Harris-Segal map, tensored with the identification $H^1_{\text{ét}}(F,\mu_m)\cong F^\times/m$, and both maps ∂ amount to the reduction mod m of the valuation map $F^\times\to\mathbb{Z}$.

Theorem 3.2. *Let F be a field containing $1/\ell$. If $\ell = 2$ we suppose that F is non-exceptional. Then for each i the Kahn map $H^1_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i}) \to K_{2i-1}(F; \mathbb{Z}/\ell^\nu)$ is an injection, split by the first étale Chern class.*

The Kahn maps are compatible with change of coefficients. Hence it induces maps $H^1_{\text{ét}}(F, \mathbb{Z}_\ell(i)) \to K_{2i-1}(F; \mathbb{Z}_\ell)$ and $H^1_{\text{ét}}(F, \mathbb{Z}/\ell^\infty(i)) \to K_{2i-1}(F; \mathbb{Z}/\ell^\infty)$.

Proof. When ℓ is odd (or $\ell = 2$ and $\sqrt{-1} \in F$), the proof that the Kahn map splits the étale Chern class is given in [Ka1], and is essentially a reorganization of Soulé's proof in [So] that the first étale Chern class is a surjection. up to factorials. (Cf. [DF]). When $\ell = 2$ and F is non-exceptional, Kahn proves in [Ka2] that this map is a split injection. □

Corollary 3.3. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F, with $1/\ell \in \mathcal{O}_S$. If $\ell = 2$, assume that F is non-exceptional. Then the Kahn maps for F induce injections $H^1_{\text{ét}}(\mathcal{O}_S, \mu_{\ell^\nu}^{\otimes i}) \to K_{2i-1}(\mathcal{O}_S; \mathbb{Z}/\ell^\nu)$, split by the first étale Chern class.*

Proof. Since $H^1_{\text{ét}}(\mathcal{O}_S, \mu_{\ell^\nu}^{\otimes i})$ is the kernel of $H^1_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i}) \to \oplus_\wp H^0_{\text{ét}}(k(\wp), \mu_m^{\otimes i-1})$, and $K_{2i-1}(\mathcal{O}_S; \mathbb{Z}/\ell^\nu)$ is the kernel of $K_{2i-1}(F; \mathbb{Z}/\ell^\nu) \to \oplus_\wp K_{2i-2}(k(\wp); \mathbb{Z}/\ell^\nu)$ by 1.6, this follows formally from 3.1.1. □

Example 3.4. If F is a number field, the first étale Chern class detects the torsionfree part of $K_{2i-1}(\mathcal{O}_F) = K_{2i-1}(F)$ described in 1.5. In fact, it induces isomorphisms $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell \cong K^{\text{ét}}_{2i-1}(\mathcal{O}_S; \mathbb{Q}_\ell) \cong H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i))$.

To see this, choose S to contain all places over some odd prime ℓ. Then $1/\ell \in \mathcal{O}_S$, and $K_{2i-1}(\mathcal{O}_S) \cong K_{2i-1}(F)$. A theorem of Tate states that

$$\text{rank } H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i)) - \text{rank } H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = \begin{cases} r_2, & i \text{ even;} \\ r_1 + r_2, & i \text{ odd.} \end{cases}$$

We will see in 3.8 below that $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = 0$. Comparing with 1.5, we see that the source and target of the first étale Chern class

$$K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell \to K^{\text{ét}}_{2i-1}(\mathcal{O}_S; \mathbb{Z}_\ell) \cong H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i))$$

have the same rank. By 3.2, this map is a split surjection (split by the Kahn map), whence the claim.

The second étale Chern class is constructed in a similar fashion. Assuming that ℓ is odd, or that $\ell = 2$ and F is non-exceptional, so that the e-invariant splits by 2.5, then for $i \geq 1$ there is also a canonical map

$$K_{2i}(F; \mathbb{Z}/\ell^\nu) \to H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1}),$$

called the *second étale Chern class*. It is the composition of the map to the étale K-group $K^{\text{ét}}_{2i}(F; \mathbb{Z}/\ell^\nu)$, or rather to the kernel of the edge map $K^{\text{ét}}_{2i}(F; \mathbb{Z}/\ell^\nu) \to H^0_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1})$, followed by the secondary edge map in the Atiyah-Hirzebruch spectral sequence for étale K-theory [DF].

Even if $\ell = 2$ and F is exceptional, this composition will define a family of second étale Chern classes $K_{2i}(F) \to H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1})$ and hence $K_{2i}(F) \to H^2_{\text{ét}}(F, \mathbb{Z}_\ell(i+1))$. This is because the e-invariant (2.1) factors through the map $K_{2i}(F; \mathbb{Z}/\ell^\nu) \to K_{2i-1}(F)$.

For $i = 1$, the second étale Chern class $K_2(F)/m \to H^2_{\text{ét}}(F, \mu_m^{\otimes 2})$ is just Tate's map, described in 1.3; it is an isomorphism for all F by the Merkurev-Suslin theorem.

Using this case, Kahn proved in [Ka] that the transfer always induces an isomorphism $H^2_{\text{ét}}(F_\nu, \mu_{\ell^\nu}^{\otimes i})_{\Gamma_\nu} \xrightarrow{\cong} H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i})$. Here, F_ν and $\Gamma_\nu = \text{Gal}(F_\nu/F)$ are as in 3.1 above, and if $\ell = 2$ we assume that F is non-exceptional. As before, we have an isomorphism of Γ_ν-modules $H^2_{\text{ét}}(F_\nu, \mu_{\ell^\nu}^{\otimes i+1}) \cong K_2(F_\nu) \otimes \mu_{\ell^\nu}^{\otimes i-1}$.

Definition 3.5. The *Kahn map* $H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1}) \to K_{2i}(F; \mathbb{Z}/\ell^\nu)$ is the composition

$$H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1}) \xleftarrow{\cong} H^2_{\text{ét}}(F_\nu, \mu_{\ell^\nu}^{\otimes i+1})_{\Gamma_\nu} = \Big[K_2(F_\nu) \otimes \mu_{\ell^\nu}^{\otimes i-1}\Big]_{\Gamma_\nu} \xrightarrow{\text{Harris-Segal}}$$

$$\to \Big[K_2(F_\nu) \otimes K_{2i-2}(F_\nu; \mathbb{Z}/\ell^\nu)\Big]_{\Gamma_\nu} \xrightarrow{\cup} K_{2i}(F_\nu; \mathbb{Z}/\ell^\nu)_{\Gamma_\nu} \xrightarrow{\text{transfer}} K_{2i}(F; \mathbb{Z}/\ell^\nu).$$

Compatibility 3.5.1. Let F be the quotient field of a discrete valuation ring whose residue field k contains $1/\ell$. Then the first and second Kahn maps are compatible with the maps ∂, from $H^2_{\text{ét}}(F)$ to $H^1_{\text{ét}}(k)$ and from $K_{2i}(F; \mathbb{Z}/m)$ to $K_{2i-1}(k; \mathbb{Z}/m)$. The argument here is the same as for 3.1.1.

As with 3.2, the following theorem was proven in [Ka1, Ka2].

Theorem 3.6. *Let F be a field containing $1/\ell$. If $\ell = 2$ we suppose that F is non-exceptional. Then for each $i \geq 1$ the Kahn map $H^2_{\text{ét}}(F, \mu_{\ell^\nu}^{\otimes i+1}) \to K_{2i}(F; \mathbb{Z}/\ell^\nu)$ is an injection, split by the second étale Chern class.*

The Kahn map is compatible with change of coefficients. Hence it induces maps $H^2_{\text{ét}}(F, \mathbb{Z}_\ell(i+1)) \to K_{2i}(F; \mathbb{Z}_\ell)$ and $H^2_{\text{ét}}(F, \mathbb{Z}/\ell^\infty(i+1)) \to K_{2i}(F; \mathbb{Z}/\ell^\infty)$.

Corollary 3.7. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F, with $1/\ell \in \mathcal{O}_S$. If $\ell = 2$, assume that F is non-exceptional. Then for each $i > 0$, the Kahn maps induce injections $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i+1)) \to K_{2i}(\mathcal{O}_S; \mathbb{Z}_\ell)$, split by the second étale Chern class.*

Proof. Since $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i+1))$ is the kernel of $H^2_{\text{ét}}(F, \mathbb{Z}_\ell(i+1)) \to \oplus_\wp H^1_{\text{ét}}(k(\wp), \mathbb{Z}_\ell(i))$, and $K_{2i}(\mathcal{O}_S; \mathbb{Z}_\ell)$ is the kernel of $K_{2i}(F; \mathbb{Z}_\ell) \to \oplus_\wp K_{2i-1}(k(\wp); \mathbb{Z}_\ell)$, this follows formally from 3.5.1. □

Remark 3.7.1. For each ν, $H^2_{\text{ét}}(\mathcal{O}_S, \mu_{\ell^\nu}^{\otimes i+1}) \to K_{2i}(\mathcal{O}_S; \mathbb{Z}/\ell^\nu)$ is also a split surjection, essentially because the map $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i+1)) \to H^2_{\text{ét}}(\mathcal{O}_S, \mu_{\ell^\nu}^{\otimes i+1})$ is onto; see [Ka1, 5.2].

The summand $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is finite by the following calculation.

Proposition 3.8. *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F with $1/\ell \in \mathcal{O}_S$. Then for all $i \geq 2$, $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is a finite group, and $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = 0$.*

Finally, $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}/\ell^\infty(i)) = 0$ if ℓ is odd, or if $\ell = 2$ and F is totally imaginary.

Proof. If ℓ is odd or if $\ell = 2$ and F is totally imaginary, then $H^3_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i)) = 0$ by [Se], so $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}/\ell^\infty(i))$ is a quotient of $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i))$. Since $H^2_{\text{ét}}(R, \mathbb{Q}_\ell(i)) =$

$H^2_{\text{ét}}(R, \mathbb{Z}_\ell(i)) \otimes \mathbb{Q}$, it suffices to prove the first assertion for $i > 0$. But $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is a summand of $K_{2i-2}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell$ for $i \geq 2$ by 3.7, which is a finite group by theorem 1.5.

If $\ell = 2$ and F is exceptional, the usual transfer argument for $\mathcal{O}_S \subset \mathcal{O}_{S'} \subset F(\sqrt{-1})$ shows that the kernel A of $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_2(i)) \to H^2_{\text{ét}}(\mathcal{O}_{S'}, \mathbb{Z}_2(i))$ has exponent 2. Since A must inject into the finite group $H^2_{\text{ét}}(\mathcal{O}_S, \mu_2)$, A must also be finite. Hence $H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_2(i))$ is also finite, and $H^2_{\text{ét}}(R, \mathbb{Q}_\ell(i)) = H^2_{\text{ét}}(R, \mathbb{Z}_\ell(i)) \otimes \mathbb{Q} = 0$. □

Taking the direct limit over all finite S yields:

Corollary 3.8.1. *Let F be a number field. Then $H^2_{\text{ét}}(F, \mathbb{Z}/\ell^\infty(i)) = 0$ for all odd primes ℓ and all $i \geq 2$.*

Example 3.8.2. The Main Conjecture of Iwasawa Theory, proved by Mazur and Wiles [MW], implies that (for odd ℓ) the order of the finite group $H^2_{\text{ét}}(\mathbb{Z}[1/\ell], \mathbb{Z}_\ell(2k))$ is the ℓ-primary part of the numerator of $\zeta_{\mathbb{Q}}(1-2k)$. See for example [RW, Appendix A] or [KNF, 4.2 and 6.3]. Note that by Euler's formula 2.10.1 this is also the ℓ-primary part of the numerator of $B_k/2k$, where B_k is the Bernoulli number discussed in 2.10.

Real number fields 3.8.3. If $\ell = 2$, the vanishing conclusion of 3.8.1 still holds when F is totally imaginary. However, it fails when F has $r_1 > 0$ embeddings into $\mathbb{R}$:

$$H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \cong H^2(F; \mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i \geq 3 \text{ odd} \\ 0, & i \geq 2 \text{ even.} \end{cases}$$

One way to do this computation is to observe that, by 3.8, $H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ has exponent 2. Hence the Kummer sequence is:

$$0 \to H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \to H^3(\mathcal{O}_S; \mathbb{Z}/2) \to H^3(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \to 0.$$

Now plug in the values of the right two groups, which are known by Tate-Poitou duality: $H^3(\mathcal{O}_S; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$, while $H^3(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ is: $(\mathbb{Z}/2)^{r_1}$ for i even, and 0 for i odd.

3.8.4. Suppose that F is totally real ($r_2 = 0$), and set $w_i = w_i^{(\ell)}(F)$. If $i > 0$ is even then $H^1(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}/w_i$; this group is finite. If i is odd then $H^1(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}_\ell^{r_1} \oplus \mathbb{Z}/w_i$; this is infinite. These facts may be obtained by combining the rank calculations of 3.4 and 3.8 with (2.1) and universal coefficients.

Theorem 3.9. *For every number field F, and all i, the Adams operation ψ^k acts on $K_{2i-1}(F) \otimes \mathbb{Q}$ as multiplication by k^i.*

Proof. The case $i = 1$ is well known, so we assume that $i \geq 2$. If S contains all places over some odd prime ℓ we saw in 3.4 that $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell \cong K^{\text{ét}}_{2i-1}(\mathcal{O}_S; \mathbb{Q}_\ell) \cong H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i))$. Since this isomorphism commutes with the Adams operations, and Soulé has shown in [So3] the $\psi^k = k^i$ on $H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Q}_\ell(i))$, the same must be true on $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell = K_{2i-1}(F) \otimes \mathbb{Q}_\ell$. □

§4. Global fields of finite characteristic

A global field of finite characteristic p is a finitely generated field F of transcendence degree one over $\mathbb{F}_p$; the algebraic closure of $\mathbb{F}_p$ in F is a finite field $\mathbb{F}_q$ of characteristic p. It is classical (see [Hart, I.6]) that there is a unique smooth projective curve X over $\mathbb{F}_q$ whose function field is F. If S is a nonempty set of closed points of X, then $X - S$ is affine; we call the coordinate ring R of $X - S$ the ring of *S-integers in F*. In this section, we discuss the K-theory of F, X and the rings of S-integers of F.

The group $K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$ is finitely generated of rank two by a theorem of Weil. In fact, there is a finite group $J(X)$ such that $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus J(X)$. For K_1 and $K_2(X)$, the localization sequence of Quillen [Q2] implies that there is an exact sequence

$$0 \to K_2(X) \to K_2(F) \xrightarrow{\partial} \oplus_{x \in X} k(x)^\times \to K_1(X) \to \mathbb{F}_q^\times \to 0.$$

By classical Weil reciprocity, the cokernel of ∂ is $\mathbb{F}_q$, so $K_1(X) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Bass and Tate proved in [BT] that the kernel $K_2(X)$ of ∂ is finite of order prime to p. This establishes the low dimensional cases of the following theorem, first proven by Harder [Har], using the method pioneered by Borel [Bo].

Theorem 4.1. *Let X be a smooth projective curve over a finite field of characteristic p. For $n \geq 1$, the groups $K_n(X)$ are finite groups of order prime to p.*

Proof. Tate proved that $K_n^M(F) = 0$ for all $n \geq 3$. By Geisser and Levine's theorem [GL], the Quillen groups $K_n(F)$ are uniquely p-divisible for $n \geq 3$. For every closed point $x \in X$, the groups $K_n(x)$ are finite of order prime to p ($n > 0$) because $k(x)$ is a finite field extension of $\mathbb{F}_q$. From the localization sequence

$$\oplus_{x \in X} K_n(x) \to K_n(X) \to K_n(F) \to \oplus_{x \in X} K_{n-1}(x)$$

and a diagram chase, it follows that $K_n(X)$ is uniquely p-divisible. Now Quillen proved in [GQ] that the groups $K_n(X)$ are finitely generated abelian groups. A second diagram chase shows that the groups $K_n(X)$ must be finite. □

Corollary 4.2. *If R is the ring of S-integers in $F = \mathbb{F}_q(X)$ (and $S \neq \emptyset$) then:*
a) $K_1(R) \cong R^\times \cong \mathbb{F}_q^\times \times \mathbb{Z}^s$, $|S| = s + 1$;
b) For $n \geq 2$, $K_n(R)$ is a finite group of order prime to p.

Proof. Classically, $K_1(R) = R^\times \oplus SK_1(R)$ and the units of R are well known. The computation that $SK_1(R) = 0$ is proven in [BMS]. The rest follows from the localization sequence $K_n(X) \to K_n(X') \to \oplus_{x \in S} K_{n-1}(x)$. □

The e-invariant 4.3. The targets of the e-invariant of X and F are the same groups as for $\mathbb{F}_q$, because every root of unity is algebraic over $\mathbb{F}_q$. Hence the inclusions of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$ in $K_{2i-1}(X)$ and $K_{2i-1}(F)$ are split by the e-invariant, and this group is the Harris-Segal summand.

The inverse limit of the finite curves $X_\nu = X \times \mathrm{Spec}(\mathbb{F}_{q^\nu})$ is the curve $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ over the algebraic closure $\bar{\mathbb{F}}_q$. To understand $K_n(X)$ for $n > 1$, it is useful to know not only what the groups $K_n(\bar{X})$ are, but how the (geometric) Frobenius $\varphi : x \mapsto x^q$ acts on them.

Classically, $K_0(\bar{X}) = \mathbb{Z}\oplus\mathbb{Z}\oplus J(\bar{X})$, where $J(\bar{X})$ is the group of points on the Jacobian variety over $\bar{\mathbb{F}}_q$; it is a divisible torsion group. If $\ell \neq p$, the ℓ-primary torsion subgroup $J(\bar{X})_\ell$ of $J(\bar{X})$ is isomorphic to the abelian group $(\mathbb{Z}/\ell^\infty)^{2g}$. The group $J(\bar{X})$ may or may not have p-torsion. For example, if X is an elliptic curve then the p-torsion in $J(\bar{X})$ is either 0 or $\mathbb{Z}/p^\infty$, depending on whether or not X is supersingular (see [Hart, Ex. IV.4.15]). Note that the localization $J(\bar{X})[1/p]$ is the direct sum over all $\ell \neq p$ of the ℓ-primary groups $J(\bar{X})_\ell$.

Next, recall that the group of units $\bar{\mathbb{F}}_q^\times$ may be identified with the group μ of all roots of unity in $\bar{\mathbb{F}}_q$; its underlying abelian group is isomorphic to $\mathbb{Q}/\mathbb{Z}[1/p]$. Passing to the direct limit of the $K_1(X_\nu)$ yields $K_1(\bar{X}) \cong \mu \oplus \mu$.

For $n \geq 1$, the groups $K_n(\bar{X})$ are all torsion groups, of order prime to p, because this is true of each $K_n(X_\nu)$ by 4.1. The following theorem determines the abelian group structure of the $K_n(\bar{X})$ as well as the action of the Galois group on them. It depends upon Suslin's theorem (see [Su4]) that for $i \geq 1$ and $\ell \neq p$ the groups $H^n_M(\bar{X}, \mathbb{Z}/\ell^\infty(i))$ equal the groups $H^n_{\text{ét}}(\bar{X}, \mathbb{Z}/\ell^\infty(i))$.

Theorem 4.4. *Let X be a smooth projective curve over $\mathbb{F}_q$. Then for all $n \geq 0$ we have isomorphisms of* $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$*-modules:*

$$K_n(\bar{X}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X}), & n = 0 \\ \mu(i) \oplus \mu(i), & n = 2i-1 > 0 \\ J(\bar{X})[1/p](i), & n = 2i > 0. \end{cases}$$

For $\ell \neq p$, the ℓ-primary subgroup of $K_{n-1}(X)$ is isomorphic to $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$, $n > 0$, whose Galois module structure is given by:

$$K_n(\bar{X}; \mathbb{Z}/\ell^\infty) \cong \begin{cases} \mathbb{Z}/\ell^\infty(i) \oplus \mathbb{Z}/\ell^\infty(i), & n = 2i \geq 0 \\ J(\bar{X})_\ell(i-1), & n = 2i-1 > 0. \end{cases}$$

Proof. Since the groups $K_n(\bar{X})$ are torsion for all $n > 0$, the universal coefficient theorem shows that $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$ is isomorphic to the ℓ-primary subgroup of $K_{n-1}(\bar{X})$. Thus we only need to determine the Galois modules $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. For $n = 0, 1, 2$ they may be read off from the above discussion. For $n > 2$ we consider the motivic spectral sequence (0.5); by Suslin's theorem, the terms $E_2^{p,q}$ vanish for $q < 0$ unless $p = q, q+1, q+2$. There is no room for differentials, so the spectral sequence degenerates at E_2 to yield the groups $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. There are no extension issues because the edge maps are the e-invariants $K_{2i}(X; \mathbb{Z}/\ell^\infty) \to H^0_{\text{ét}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/\ell^\infty(i)$ of 4.3, and are therefore split surjections. Finally, we note that as Galois modules we have $H^1_{\text{ét}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)$, and (by Poincaré Duality [Mi, V.2]) $H^2_{\text{ét}}(\bar{X}, \mathbb{Z}/\ell^\infty(i+1)) \cong \mathbb{Z}/\ell^\infty(i)$. □

Passing to invariants under the group $G = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, there is a natural map from $K_n(X)$ to $K_n(\bar{X})^G$. For odd n, we see from 4.4 and 2.4 that $K_{2i-1}(\bar{X})^G \cong \mathbb{Z}/(q^i-1) \oplus \mathbb{Z}/(q^i-1)$; for even n, we have the less concrete description $K_{2i}(\bar{X})^G \cong J(\bar{X})[1/p](i)^G$. One way of studying this group is to consider the action of the algebraic Frobenius φ^* (induced by φ^{-1}) on cohomology.

Example 4.5. φ^* acts trivially on $H^0_{\text{ét}}(\bar{X},\mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$. It acts as q^{-i} on the twisted groups $H^0_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(i))$ and $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(i+1)$.

Weil's proof in 1948 of the Riemann Hypothesis for Curves implies that the eigenvalues of φ^* acting on $H^1_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(i))$ have absolute value $q^{1/2-i}$.

Since $H^n_{\text{ét}}(X,\mathbb{Q}_\ell(i)) \cong H^n_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(i))^G$, a perusal of these cases shows that we have $H^n_{\text{ét}}(X,\mathbb{Q}_\ell(i)) = 0$ except when (n,i) is $(0,0)$ or $(2,1)$.

For any G-module M, we have an exact sequence [WK, 6.1.4]

$$0 \to M^G \to M \xrightarrow{\varphi^*-1} M \to H^1(G,M) \to 0. \tag{4.5.1}$$

The case $i = 1$ of the following result reproduces Weil's theorem that the ℓ-primary torsion part of the Picard group of X is $J(\bar{X})^G_\ell$.

Lemma 4.6. *For a smooth projective curve X over $\mathbb{F}_q$, $\ell \not| q$ and $i \geq 2$ we have:*

(1) $H^{n+1}_{\text{ét}}(X,\mathbb{Z}_\ell(i)) \cong H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong H^n_{\text{ét}}(\bar{X},\mathbb{Z}/\ell^\infty(i))^G$ *for all n;*
(2) $H^0_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_i^{(\ell)}(F)$;
(3) $H^1_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$;
(4) $H^2_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_{i-1}^{(\ell)}(F)$; *and*
(5) $H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) = 0$ *for all $n \geq 3$.*

Proof. Since $i \geq 2$, we see from 4.5 that $H^n_{\text{ét}}(X,\mathbb{Q}_\ell(i)) = 0$. Since $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathbb{Z}/\ell^\infty$, this yields $H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong H^{n+1}_{\text{ét}}(X,\mathbb{Z}_\ell(i))$ for all n.

Since each $H^n = H^n_{\text{ét}}(\bar{X},\mathbb{Z}/\ell^\infty(i))$ is a quotient of $H^n_{\text{ét}}(\bar{X},\mathbb{Q}_\ell(i))$, φ^*-1 is a surjection, *i.e.*, $H^1(G,H^n) = 0$. Since $H^n(G,-) = 0$ for $n > 1$, the Leray spectral sequence for $\bar{X} \to X$ collapses for $i > 1$ to yield exact sequences

$$0 \to H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \to H^n_{\text{ét}}(\bar{X},\mathbb{Z}/\ell^\infty(i)) \xrightarrow{\varphi^*-1} H^n_{\text{ét}}(\bar{X},\mathbb{Z}/\ell^\infty(i)) \to 0. \tag{4.6.1}$$

In particular, $H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) = 0$ for $n>2$. Since $H^2_{\text{ét}}(\bar{X},\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/\ell^\infty(i-1)$ this yields $H^2_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/\ell^\infty(i-1)^G = \mathbb{Z}/w_{i-1}$. We also see that $H^1_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i))$ is the group of invariants of the Frobenius, *i.e.*, $J(X)_\ell(i-1)^{\varphi^*}$. □

Given the calculation of $K_n(\bar{X})^G$ in 4.4 and the calculation of $H^n_{\text{ét}}(X,\mathbb{Z}/\ell^\infty(i))$ in 4.6, we see that the natural map $K_n(X) \to K_n(\bar{X})^G$ is a surjection, split by the Kahn maps 3.2 and 3.6. Thus the real content of the following theorem is that $K_n(X) \to K_n(\bar{X})^G$ is an isomorphism.

Theorem 4.7. *Let X be the smooth projective curve corresponding to a global field F over $\mathbb{F}_q$. Then $K_0(X) = \mathbb{Z} \oplus \text{Pic}(X)$, and the finite groups $K_n(X)$ for $n > 0$ are given by:*

$$K_n(X) \cong K_n(\bar{X})^G \cong \begin{cases} K_n(\mathbb{F}_q) \oplus K_n(\mathbb{F}_q), & n \text{ odd,} \\ \bigoplus_{\ell \neq p} J(\bar{X})_\ell(i)^G, & n = 2i \text{ even.} \end{cases}$$

Proof. We may assume that $n \neq 0$, so that the groups $K_n(X)$ are finite by 4.1. It suffices to calculate the ℓ-primary part $K_{n+1}(X;\mathbb{Z}/\ell^\infty)$ of $K_n(X)$. But this follows from the motivic spectral sequence (0.5), which degenerates by 4.6. □

The Zeta Function 4.8. We can relate the orders of the K-groups of the curve X to values of the zeta function $\zeta_X(s)$. By definition, $\zeta_X(s) = Z(X, q^{-s})$, where

$$Z(X,t) = \exp\Big(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})|\,\frac{t^n}{n}\Big).$$

Weil proved that $Z(X,t) = P(t)/(1-t)(1-qt)$ for every smooth projective curve X, where $P(t) \in \mathbb{Z}[t]$ is a polynomial of degree $2 \cdot \mathrm{genus}(X)$ with all roots of absolute value $1/\sqrt{q}$. This formula is a restatement of Weil's proof of the Riemann Hypothesis for X (4.5 above), given Grothendieck's formula $P(t) = \det(1 - \varphi^* t)$, where φ^* is regarded as an endomorphism of $H^1_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell)$. Note that by 4.5 the action of φ^* on $H^0_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell)$ has $\det(1 - \varphi^* t) = (1-t)$, and the action on $H^2_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell)$ has $\det(1 - \varphi^* t) = (1 - qt)$.

Here is application of theorem 4.7, which goes back to Thomason (see [Th, (4.7)] and [Li3]). Let $\#A$ denote the order of a finite abelian group A.

Corollary 4.9. *If X is a smooth projective curve over $\mathbb{F}_q$ then for all $i \geq 2$,*

$$\frac{\#K_{2i-2}(X) \cdot \#K_{2i-3}(\mathbb{F}_q)}{\#K_{2i-1}(\mathbb{F}_q) \cdot \#K_{2i-3}(X)} = \prod_\ell \frac{\#H^2_{\text{ét}}(X; \mathbb{Z}_\ell(i))}{\#H^1_{\text{ét}}(X; \mathbb{Z}_\ell(i)) \cdot \#H^3_{\text{ét}}(X; \mathbb{Z}_\ell(i))} = \big|\zeta_X(1-i)\big|.$$

Proof. We have seen that all the groups appearing in this formula are finite. The first equality follows from 4.6 and 4.7. The second equality follows from the formula for $\zeta_X(1-i)$ in 4.8. □

Iwasawa modules 4.9. It is worth noting that the group $H^1_{\text{ét}}(X, \mathbb{Z}/\ell^\infty(i))$ is the (finite) group of invariants $M^\#(i)^{\varphi^*}$ of the ith twist of the Pontrjagin dual $M^\#$ of the *Iwasawa module* $M = M_X$. By definition M_X is the Galois group of $\hat{X}$ over $X_\infty = X \otimes_{\mathbb{F}_q} F_q(\infty)$, where the field $F_q(\infty)$ is obtained from $\mathbb{F}_q$ by adding all ℓ-primary roots of unity, and $\hat{X}$ is the maximal unramified pro-ℓ abelian cover of X_∞. It is known that the Iwasawa module M_X is a finitely generated free $\mathbb{Z}_\ell$-module, and that its dual $M^\#$ is a finite direct sum of copies of $\mathbb{Z}/\ell^\infty$ [DM, 3.22]. This viewpoint was developed in [DM], and the corresponding discussion of Iwasawa modules for number fields is in [MKH].

§5. Local Fields

Let E be a local field of residue characteristic p, with (discrete) valuation ring V and residue field $\mathbb{F}_q$. It is well known that $K_0(V) = K_0(E) = \mathbb{Z}$ and $K_1(V) = V^\times$, $K_1(E) = E^\times \cong (V^\times) \times \mathbb{Z}$, where the factor $\mathbb{Z}$ is identified with the powers $\{\pi^m\}$ of a parameter π of V. It is well known that $V^\times \cong \mu(E) \times U_1$, where $\mu(E)$ is the group of roots of unity in E (or V), and that U_1 is a free $\mathbb{Z}_p$-module.

In the equi-characteristic case, where $\mathrm{char}(E) = p$, it is well known that $V \cong \mathbb{F}_q[[\pi]]$ and $E = \mathbb{F}_q((\pi))$ [Se], so $\mu(E) = \mathbb{F}_q^\times$, and $U_1 = W(\mathbb{F}_q)$ has rank $[\mathbb{F}_q : \mathbb{F}_p]$ over $\mathbb{Z}_p = W(\mathbb{F}_p)$. The decomposition of $K_1(V) = V^\times$ is evident here. Here is a description of the abelian group structure on $K_n(V)$ for $n > 1$.

Theorem 5.1. *Let $V = \mathbb{F}_q[[\pi]]$ be the ring of integers in the local field $E = \mathbb{F}_q((\pi))$. For $n \geq 2$ there are uncountable, uniquely divisible abelian groups U_n so that*

$$K_n(V) \cong K_n(\mathbb{F}_q) \oplus U_n, \quad K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q).$$

Proof. The map $K_{n-1}(\mathbb{F}_q) \to K_n(E)$ sending x to $\{x, \pi\}$ splits the localization sequence, yielding the decomposition of $K_n(E)$. If U_n denotes the kernel of the canonical map $K_n(V) \to K_n(\mathbb{F}_q)$, then naturality yields $K_n(V) = U_n \oplus K_n(\mathbb{F}_q)$. By Gabber's rigidity theorem [Gab], U_n is uniquely ℓ-divisible for $\ell \neq p$ and $n > 0$. It suffices to show that U_n is uncountable and uniquely p-divisible when $n \geq 2$.

Tate showed that the Milnor groups $K_n^M(E)$ are uncountable, uniquely divisible for $n \geq 3$, and that the same is true for the kernel U_2 of the norm residue map $K_2(E) \to \mu(E)$; see [Ta2]. If $n \geq 2$ then $K_n^M(E)$ is a summand of the Quillen K-group $K_n(E)$ by [Su3]. On the other hand, Geisser and Levine proved in [GL] that the complementary summand is uniquely p-divisible. □

In the mixed characteristic case, when $\mathrm{char}(E) = 0$, even the structure of $V^\times$ is quite interesting. The torsionfree part U_1 is a free $\mathbb{Z}_p$-module of rank $[E : \mathbb{Q}_p]$; it is contained in $(1 + \pi V)^\times$ and injects into E by the convergent power series for $x \mapsto \ln(x)$.

The group $\mu(E)$ of roots of unity in E (or V) is identified with $(\mathbb{F}_q^*) \times \mu_{p^\infty}(E)$, where the first factor arises from Teichmüller's theorem that $V^\times \to \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)$ has a unique splitting, and $\mu_{p^\infty}(E)$ denotes the finite group of p-primary roots of unity in E. There seems to be no simple formula for the order of the cyclic p-group $\mu_{p^\infty}(E)$

For K_2, there is a norm residue symbol $K_2(E) \to \mu(E)$ and we have the following result; see [WK, III.6.6].

Moore's Theorem 5.2. *The group $K_2(E)$ is the product of a finite group, isomorphic to $\mu(E)$, and an uncountable, uniquely divisible abelian group U_2. In addition,*

$$K_2(V) \cong \mu_{p^\infty}(E) \times U_2.$$

Proof. The fact that the kernel U_2 of the norm residue map is divisible is due to C. Moore, and is given in the Appendix to [Mil]. The fact that U_2 is torsionfree (hence uniquely divisible) was proven by Tate [Ta2] when $\mathrm{char}(F) = p$, and by Merkurjev [Merk] when $\mathrm{char}(F) = 0$. □

Since the transcendence degree of E over $\mathbb{Q}$ is uncountable, it follows from Moore's theorem and the arguments in [Mil] that the Milnor K-groups $K_n^M(E)$ are uncountable, uniquely divisible abelian groups for $n \geq 3$. By [Su3], this is a summand of the Quillen K-group $K_n(E)$. As in the equicharacteristic case, $K_n(E)$ will contain an uncountable uniquely divisible summand about which we can say very little.

To understand the other factor, we typically proceed a prime at a time. This has the advantage of picking up the torsion subgroups of $K_n(E)$, and detecting the groups $K_n(V)/\ell$. For p-adic fields, the following calculation reduces the problem to the prime p.

Proposition 5.3. *If $i > 0$ there is a summand of $K_{2i-1}(V) \cong K_{2i-1}(E)$ isomorphic to $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$, detected by the e-invariant. The complementary summand is uniquely ℓ-divisible for every prime $\ell \neq p$, i.e., a $\mathbb{Z}_{(p)}$-module.*

There is also a decomposition $K_{2i}(E) \cong K_{2i}(V) \oplus K_{2i-1}(\mathbb{F}_q)$, and the group $K_{2i}(V)$ is uniquely ℓ-divisible for every prime $\ell \neq p$, i.e., a $\mathbb{Z}_{(p)}$-module.

Proof. Pick a prime ℓ. By Gabber's rigidity theorem [Gab], the groups $K_n(V; \mathbb{Z}/\ell^\nu)$ are isomorphic to $K_n(\mathbb{F}_q; \mathbb{Z}/\ell^\nu)$ for $n > 0$. Since the Bockstein spectral sequences are isomorphic, and detect all finite cyclic ℓ-primary summands of $K_n(V)$ and $K_n(\mathbb{F}_q)$ [WK, 5.9.12], it follows that $K_{2i-1}(V)$ has a cyclic summand isomorphic to $\mathbb{Z}/w_i^{(\ell)}(E)$, and that the complement is uniquely ℓ-divisible. Since $K_n(V; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$, we also see that $K_{2i}(V)$ is uniquely ℓ-divisible. As ℓ varies, we get a cyclic summand of order $w_i(E)$ in $K_{2i-1}(V)$ whose complement is a $\mathbb{Z}_{(p)}$-module.

If $x \in K_{2i-1}(V)$, the product $\{x, \pi\} \in K_{2i}(E)$ maps to the image of x in $K_{2i-1}(\mathbb{F}_q)$ under the boundary map ∂ in the localization sequence. Hence the summand of $K_{2i-1}(V)$ isomorphic to $K_{2i-1}(\mathbb{F}_q)$ lifts to a summand of $K_{2i}(E)$. This breaks the localization sequence up into split short exact sequences $0 \to K_n(V) \to K_n(E) \to K_{n-1}(\mathbb{F}_q) \to 0$. $\square$

Completed K-theory 5.4. It will be convenient to fix a prime ℓ and pass to the ℓ-adic completion $\hat{K}(R)$ of the K-theory space $K(R)$, where R is any ring. We also write $K_n(R; \mathbb{Z}_\ell)$ for $\pi_n \hat{K}(R)$. Information about these groups tells us about the groups $K_n(R, \mathbb{Z}/\ell^\nu) = \pi_n(K(R); \mathbb{Z}/\ell^\nu)$, because these groups are isomorphic to $\pi_n(\hat{K}(R); \mathbb{Z}/\ell^\nu)$ for all ν.

If the groups $K_n(R; \mathbb{Z}/\ell^\nu)$ are finite, then $K_n(R; \mathbb{Z}_\ell)$ is an extension of the Tate module of $K_{n-1}(R)$ by the ℓ-adic completion of $K_n(R)$. (The *Tate module* of an abelian group A is the inverse limit of the groups $\mathrm{Hom}(\mathbb{Z}/\ell^\nu, A)$.) For example, $K_n(\mathbb{C}; \mathbb{Z}_\ell)$ vanishes for odd n and for even n equals the Tate module $\mathbb{Z}_\ell$ of $K_{n-1}(\mathbb{C})$. If in addition the abelian groups $K_n(R)$ are finitely generated, there can be no Tate module and we have $K_n(R; \mathbb{Z}_\ell) \cong K_n(R) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \varprojlim K_n(R; \mathbb{Z}/\ell^\nu)$.

Warning 5.4.1. Even if we know $K_n(R; \mathbb{Z}_\ell)$ for all primes, we may not still be able to determine the underlying abelian group $K_n(R)$ exactly from this information. For example, consider the case $n = 1$, $R = \mathbb{Z}_p$. We know that $K_1(R; \mathbb{Z}_p) \cong (1 + pR)^\times \cong \mathbb{Z}_p$, $p \neq 2$, but this information does not even tell us that $K_1(R) \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_p$. To see why, note that the extension $0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z}_{(p)} \to 0$ doesn't split; there are no p-divisible elements in $\mathbb{Z}_p$, yet $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$ is a uniquely divisible abelian group.

We now consider the p-adic completion of $K(E)$. By 5.3, it suffices to consider the p-adic completion of $K(V)$.

Write w_i for the numbers $w_i = w_i^{(p)}(E)$, which were described in 2.9. For all i, and $\ell^\nu > w_i$, the étale cohomology group $H^1(E, \mu_{p^\nu}^{\otimes i})$ is isomorphic to $(\mathbb{Z}/p^\nu)^d \oplus \mathbb{Z}/w_i \oplus \mathbb{Z}/w_{i-1}$, $d = [E : \mathbb{Q}_p]$. By duality, the group $H^2(E, \mu_{p^\nu}^{\otimes i+1})$ is also isomorphic to $\mathbb{Z}/w_i$.

Theorem 5.5. *Let E be a local field, of degree d over $\mathbb{Q}_p$, with ring of integers V. Then for $n \geq 2$ we have:*

$$K_n(V; \mathbb{Z}_p) \cong K_n(E; \mathbb{Z}_p) \cong \left\{ \begin{array}{ll} \mathbb{Z}/w_i^{(p)}(E), & n = 2i, \\ (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_i^{(p)}(E), & n = 2i-1. \end{array} \right\}$$

Moreover, the first étale Chern classes $K_{2i-1}(E; \mathbb{Z}/p^\nu) \cong H^1(E, \mu_{p^\nu}^{\otimes i})$ are natural isomorphisms for all i and ν.

Finally, each $K_{2i}(V)$ is the direct sum of a uniquely divisible group, a divisible p-group and a subgroup isomorphic to $\mathbb{Z}/w_i^{(p)}(E)$.

Proof. If $p > 2$ the first part is proven in [HM, thm. A]. (It also follows from the spectral sequence (0.3) for E, using the Voevodsky-Rost theorem.) In this case, theorem 3.2 and a count shows that the étale Chern classes $H^1_{\text{ét}}(E; \mu_{p^\nu}^{\otimes i}) \to K_{2i-1}(E; \mathbb{Z}/p^\nu)$ are isomorphisms. If $p = 2$ this is proven in [RW, 1.12]; surprisingly, this implies that the Harris-Segal maps and Kahn maps are even defined when E is an exceptional 2-adic field.

Now fix i and set $w_i = w_i^{(p)}(E)$. Since the Tate module of any abelian group is torsionfree, and $K_{2i}(E; \mathbb{Z}_p)$ is finite, we see that the Tate module of $K_{2i-1}(E)$ vanishes and the p-adic completion of $K_{2i}(E)$ is $\mathbb{Z}/w_i$. Since this is also the completion of the $\mathbb{Z}_{(p)}$-module $K_{2i}(V)$ by 5.3, the decomposition follows from the structure of $\mathbb{Z}_{(p)}$-modules. (This decomposition was first observed in [Ka1, 6.2].) $\square$

Remark 5.5.1. The fact that these groups were finitely generated $\mathbb{Z}_p$-modules of rank d was first obtained by Wagoner in [Wg], modulo the identification in [Pa] of Wagoner's continuous K-groups with $K_*(E; \mathbb{Z}/p)$.

Unfortunately, I do not know how to reconstruct the "integral" homotopy groups $K_n(V)$ from the information in 5.5. Any of the $\mathbb{Z}_p$'s in $K_{2i-1}(V; \mathbb{Z}_p)$ could come from either a $\mathbb{Z}_{(p)}$ in $K_{2i-1}(V)$ or a $\mathbb{Z}/p^\infty$ in $K_{2i-2}(V)$. Here are some cases when I can show that they come from torsionfree elements; I do not know any example where a $\mathbb{Z}/p^\infty$ appears.

Corollary 5.6. *$K_3(V)$ contains a torsionfree subgroup isomorphic to $\mathbb{Z}_{(p)}^d$, whose p-adic completion is isomorphic to $K_3(V; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$.*

Proof. Combine 5.5 with Moore's theorem 5.2 and 5.3. $\square$

I doubt that the extension $0 \to \mathbb{Z}_{(p)}^d \to K_3(V) \to U_3 \to 0$ splits.

Example 5.7. If $k > 0$, $K_{4k+1}(\mathbb{Z}_2)$ contains a subgroup T_k isomorphic to $\mathbb{Z}_{(2)} \times \mathbb{Z}/w_i$, and the quotient $K_{4k+1}(\mathbb{Z}_2)/T_k$ is uniquely divisible. (By 2.9, $w_i = 2(2^{2k+1}-1)$.)

This follows from Rognes' theorem [R1, 4.13] that the map from $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2)$ to $K_{4k+1}(\mathbb{Z}_2; \mathbb{Z}_2)$ is an isomorphism for all $k > 1$. (The information about the torsion subgroups, missing in [R1], follows from [RW].) Since this map factors through $K_{4k+1}(\mathbb{Z}_2)$, the assertion follows.

Example 5.8. Let F be a totally imaginary number field of degree $d = 2r_2$ over $\mathbb{Q}$, and let $E_1, ..., E_s$ be the completions of F at the prime ideals over p. There is a subgroup of $K_{2i-1}(F)$ isomorphic to $\mathbb{Z}^{r_2}$ by theorem 1.5; its image in $\oplus K_{2i-1}(E_j)$ is a subgroup of rank at most r_2, while $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$ has rank $d = \sum [E_j : \mathbb{Q}_p]$. So these subgroups of $K_{2i-1}(E_j)$ can account for at most half of the torsionfree part of $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$.

Example 5.9. Suppose that F is a totally real number field, of degree $d = r_1$ over $\mathbb{Q}$, and let $E_1, ..., E_s$ be the completions of F at the prime ideals over p. For $k > 0$, there is a subgroup of $K_{4k+1}(F)$ isomorphic to $\mathbb{Z}^d$ by theorem 1.5; its image in $\oplus K_{4k+1}(E_j)$ is a subgroup of rank d, while $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$ has rank $d = \sum [E_j : \mathbb{Q}_p]$. However, this does not imply that the p-adic completion $\mathbb{Z}_p^d$ of the subgroup injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$. Implications like this are related to Leopoldt's conjecture.

Leopoldt's conjecture states that the torsionfree part $\mathbb{Z}_p^{d-1}$ of $(\mathcal{O}_F)^\times \otimes \mathbb{Z}_p$ injects into the torsionfree part $\mathbb{Z}_p^d$ of $\prod_{j=1}^s \mathcal{O}_{E_j}^\times$; see [Wash, 5.31]. This conjecture has been proven when F is an abelian extension of $\mathbb{Q}$; see [Wash, 5.32].

When F is a totally real abelian extension of $\mathbb{Q}$, and p is a regular prime, Soulé shows in [So, 3.1, 3.7] that the torsion free part $\mathbb{Z}_p^d$ of $K_{4k+1}(F) \otimes \mathbb{Z}_p$ injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$, because the cokernel is determined by the Leopoldt p-adic L-function $L_p(F, \omega^{2k}, 2k+1)$, which is a p-adic unit in this favorable scenario. Therefore in this case we also have a summand $\mathbb{Z}_{(p)}^d$ in each of the groups $K_{4k+1}(E_j)$.

We conclude with a description of the topological type of $\hat{K}(V)$ and $\hat{K}(E)$, when p is odd. Recall that $F\Psi^k$ denotes the homotopy fiber of $\Psi^k - 1 : \mathbb{Z} \times BU \to BU$. Since $\Psi^k = k^i$ on $\pi_{2i}(BU) = \mathbb{Z}$ for $i > 0$, and the other homotopy group of BU vanish, we see that $\pi_{2i-1} F\Psi^k \cong \mathbb{Z}/(k^i - 1)$, and that all even homotopy groups of $F\Psi^k$ vanish, except for $\pi_0(F\Psi^k) = \mathbb{Z}$.

Theorem 5.10. ([HM, thm.D]) *Let E be a local field, of degree d over $\mathbb{Q}_p$, with p odd. Then after p-completion, there is a number k (given below) so that*

$$\hat{K}(V) \simeq SU \times U^{d-1} \times F\Psi^k \times BF\Psi^k, \qquad \hat{K}(E) \simeq U^d \times F\Psi^k \times BF\Psi^k.$$

The number k is defined as follows. Set $r = [E(\mu_p) : E]$, and let p^a be the number of p-primary roots of unity in $E(\mu_p)$. If r is a topological generator of $\mathbb{Z}_p^\times$, then $k = r^n$, $n = p^{a-1}(p-1)/r$. It is an easy exercise, left to the reader, to check that $\pi_{2i-1} F\Psi^k \cong \mathbb{Z}_p/(k^i - 1)$ is $\mathbb{Z}/w_i$ for all i.

§6. Number fields at primes where $cd = 2$

In this section we quickly obtain a cohomological description of the odd torsion in the K-groups of a number field, and also the 2-primary torsion in the K-groups of a totally imaginary number field. These are the cases where $cd_\ell(\mathcal{O}_S) = 2$, which forces the motivic spectral sequence (0.5) to degenerate completely.

The following trick allows us to describe the torsion subgroup of the groups $K_n(R)$. Recall that the notation $A\{\ell\}$ denotes the ℓ-primary subgroup of an abelian group A.

Lemma 6.1. *For a given prime ℓ, ring R and integer n, suppose that $K_n(R)$ is a finite group, and that $K_{n-1}(R)$ is a finitely generated group. Then $K_n(R)\{\ell\} \cong K_n(R;\mathbb{Z}_\ell)$ and $K_{n-1}(R)\{\ell\} \cong K_n(R;\mathbb{Z}/\ell^\infty)$.*

Proof. For large values of ν, the finite group $K_n(R;\mathbb{Z}/\ell^\nu)$ is the sum of $K_n(R)\{\ell\}$ and $K_{n-1}(R)\{\ell\}$. The transition from coefficients $\mathbb{Z}/\ell^\nu$ to $\mathbb{Z}/\ell^{\nu-1}$ (resp., to $\mathbb{Z}/\ell^{\nu+1}$) is multiplication by 1 and ℓ (resp., by ℓ and 1) on the two summands. Taking the inverse limit (resp., direct limit) yields the groups $K_n(R;\mathbb{Z}_\ell)$ and $K_n(R;\mathbb{Z}/\ell^\infty)$, respectively. □

Example 6.1.1. By 1.6, the lemma applies to a ring $\mathcal{O}_S$ of integers in a number field F, with n even. For example, theorem 1.3 says that $K_2(\mathcal{O}_S)\{\ell\} = K_2(\mathcal{O}_S;\mathbb{Z}_\ell) \cong H^2_{\text{ét}}(\mathcal{O}_F[1/\ell],\mathbb{Z}_\ell(2))$, and of course $K_1(\mathcal{O}_S)\{\ell\} = K_2(\mathcal{O}_S;\mathbb{Z}/\ell^\infty)$ is the group $\mathbb{Z}/w_1^{(\ell)}(F)$ of ℓ-primary roots of unity in F.

We now turn to the odd torsion in the K-groups of a number field. The ℓ-primary torsion is described by the following result, which which is based on [RW] and uses the Voevodsky-Rost theorem. The notation $A_{(\ell)}$ will denote the localization of an abelian group A at the prime ℓ.

Theorem 6.2. *Fix an odd prime ℓ. Let F be a number field, and let $\mathcal{O}_S$ be a ring of integers in F. If $R = \mathcal{O}_S[1/\ell]$, then for all $n \geq 2$:*

$$K_n(\mathcal{O}_S)_{(\ell)} \cong \begin{cases} H^2_{\text{ét}}(R;\mathbb{Z}_\ell(i+1)) & \text{for } n = 2i > 0; \\ \mathbb{Z}_{(\ell)}^{r_2} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ even}; \\ \mathbb{Z}_{(\ell)}^{r_2+r_1} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ odd}. \end{cases}$$

Proof. By 1.6 we may replace $\mathcal{O}_S$ by R without changing the ℓ-primary torsion. By 6.1 and 1.5, it suffices to show that $K_{2i}(R;\mathbb{Z}_\ell) \cong H^2_{\text{ét}}(R;\mathbb{Z}_\ell(i+1))$ and $K_{2i}(R;\mathbb{Z}/\ell^\infty) \cong \mathbb{Z}/w_i^{(\ell)}(F)$. Note that the formulas for $K_0(\mathcal{O}_S)$ and $K_1(\mathcal{O}_S)$ are different; see (1.2.1).

If F is a number field and $\ell \neq 2$, the étale ℓ-cohomological dimension of F (and of R) is 2. Since $H^2_{\text{ét}}(R;\mathbb{Z}/\ell^\infty(i)) = 0$ by 3.8.1, the Voevodsky-Rost theorem implies that the motivic spectral sequence (0.5) has only two nonzero diagonals, except in total degree zero, and collapses at E_2. This gives

$$K_n(\mathcal{O}_S;\mathbb{Z}/\ell^\infty) \cong \begin{cases} H^0(R;\mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i \geq 2, \\ H^1(R;\mathbb{Z}/\ell^\infty(i)) & \text{for } n = 2i-1 \geq 1. \end{cases} \tag{6.2.1}$$

The description of $K_{2i-1}(\mathcal{O}_S)\{\ell\}$ follows from 6.1 and 1.5.

The same argument works for coefficients $\mathbb{Z}_\ell$; for $i>0$ we have $H^n_{\text{ét}}(R,\mathbb{Z}_\ell(i))=0$ for $n\neq 1,2$, so the spectral sequence degenerates to yield $K_{2i}(R;\mathbb{Z}_\ell)\cong H^2_{\text{ét}}(R,\mathbb{Z}_\ell(i))$. (This is a finite group by 3.8.) The description of $K_{2i}(R)\{\ell\}$ follows from 6.1 and 1.5. □

Because $H^2_{\text{ét}}(R,\mathbb{Z}_\ell(i+1))/\ell\cong H^2_{\text{ét}}(R,\mu_\ell^{\otimes i+1})$, we immediately deduce:

Corollary 6.3. *For all odd ℓ and $i>0$, $K_{2i}(\mathcal{O}_S)/\ell\cong H^2_{\text{ét}}(\mathcal{O}_S[1/\ell],\mu_\ell^{\otimes i+1})$.*

Remark 6.4. Similarly, the mod-ℓ spectral sequence (0.3) collapses to yield the K-theory of $\mathcal{O}_S$ with coefficients $\mathbb{Z}/\ell$, ℓ odd. For example, if $\mathcal{O}_S$ contains a primitive ℓth root of unity and $1/\ell$ then $H^1(\mathcal{O}_S;\mu_\ell^{\otimes i})\cong\mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell}\oplus\ {}_\ell\operatorname{Pic}(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S;\mu_\ell^{\otimes i})\cong\operatorname{Pic}(\mathcal{O}_S)/\ell\oplus\ {}_\ell\operatorname{Br}(\mathcal{O}_S)$ for all i, so

$$K_n(\mathcal{O}_S;\mathbb{Z}/\ell)\cong\begin{cases}\mathbb{Z}/\ell\oplus\operatorname{Pic}(\mathcal{O}_S)/\ell, & n=0\\ \mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell}\oplus\ {}_\ell\operatorname{Pic}(\mathcal{O}_S) & \text{for } n=2i-1\geq 1,\\ \mathbb{Z}/\ell\oplus\operatorname{Pic}(\mathcal{O}_S)/\ell\oplus\ {}_\ell\operatorname{Br}(\mathcal{O}_S) & \text{for } n=2i\geq 2,\end{cases}$$

The $\mathbb{Z}/\ell$ summands in degrees $2i$ are generated by the powers β^i of the Bott element $\beta\in K_2(\mathcal{O}_S;\mathbb{Z}/\ell)$ (see 2.5.1). In fact, $K_*(\mathcal{O}_S;\mathbb{Z}/\ell)$ is free as a graded $\mathbb{Z}[\beta]$-module on $K_0(\mathcal{O}_S;\mathbb{Z}/\ell)$, $K_1(\mathcal{O}_S;\mathbb{Z}/\ell)$ and ${}_\ell\operatorname{Br}(\mathcal{O}_S)\in K_2(\mathcal{O}_S;\mathbb{Z}/\ell)$; this is immediate from the multiplicative properties of (0.3).

When F is totally imaginary, we have a complete description of $K_*(\mathcal{O}_S)$. The 2-primary torsion was first calculated in [RW]; the odd torsion comes from theorem 6.2.

Theorem 6.5. *Let F be a totally imaginary number field, and let $\mathcal{O}_S$ be the ring of S-integers in F for some set S of finite places. Then for all $n\geq 2$:*

$$K_n(\mathcal{O}_S)\cong\begin{cases}\mathbb{Z}\oplus\operatorname{Pic}(\mathcal{O}_S), & \text{for } n=0;\\ \mathbb{Z}^{r_2+|S|-1}\oplus\mathbb{Z}/w_1, & \text{for } n=1;\\ \oplus_\ell H^2_{\text{ét}}(\mathcal{O}_S[1/\ell];\mathbb{Z}_\ell(i+1)) & \text{for } n=2i\geq 2;\\ \mathbb{Z}^{r_2}\oplus\mathbb{Z}/w_i & \text{for } n=2i-1\geq 3.\end{cases}$$

Proof. The case $n=1$ comes from (1.2.1), and the odd torsion comes from 6.2, so it suffices to check the 2-primary torsion. This does not change if we replace $\mathcal{O}_S$ by $R=\mathcal{O}_S[1/2]$, by 1.6. By 6.1 and 1.5, it suffices to show that $K_{2i}(R;\mathbb{Z}_2)\cong H^2_{\text{ét}}(R;\mathbb{Z}_2(i+1))$ and $K_{2i}(R;\mathbb{Z}/2^\infty)\cong\mathbb{Z}/w_i^{(2)}(F)$.

Consider the mod 2^∞ motivic spectral sequence (0.5) for the ring R, converging to $K_*(R;\mathbb{Z}/2^\infty)$. It is well known that $cd_2(R)=2$, and $H^2_{\text{ét}}(R;\mathbb{Z}/2^\infty(i))=0$ by 3.8.1. Hence the spectral sequence collapses; except in total degree zero, the E_2-terms are concentrated on the two diagonal lines where $p=q$, $p=q+1$. This gives

$$K_n(R;\mathbb{Z}/2^\infty)\cong\begin{cases}H^0(R;\mathbb{Z}/2^\infty(i))=\mathbb{Z}/w_i^{(2)}(F) & \text{for } n=2i\geq 0,\\ H^1(R;\mathbb{Z}/2^\infty(i)) & \text{for } n=2i-1\geq 1.\end{cases}$$

The description of $K_{2i-1}(R)\{2\}$ follows from 6.1 and 1.5.

The same argument works for coefficients $\mathbb{Z}_2$; for $i>0$ we have $H^n_{\text{ét}}(R,\mathbb{Z}_2(i))=0$ for $n\neq 1,2$, so (0.5) degenerates to yield $K_{2i}(R;\mathbb{Z}_2)\cong H^2_{\text{ét}}(R,\mathbb{Z}_2(i))$. (This is a finite group by 3.8). The description of $K_{2i}(R)\{2\}$ follows from 6.1 and 1.5. □

Example 6.6. Let F be a number field containing a primitive ℓth root of unity, and let S be the set of primes over ℓ in $\mathcal{O}_F$. If t is the rank of $\mathrm{Pic}(R)/\ell$, then $H^2_{\text{ét}}(R, \mathbb{Z}_\ell(i))/\ell \cong H^2_{\text{ét}}(R, \mu_\ell^{\otimes i}) \cong H^2_{\text{ét}}(R, \mu_\ell) \otimes \mu_\ell^{\otimes i-1}$ has rank $t + |S| - 1$ by (1.3.1). By 6.5, the ℓ-primary subgroup of $K_{2i}(\mathcal{O}_S)$ has $t + |S| - 1$ nonzero summands for each $i \geq 2$.

Example 6.7. If $\ell \neq 2$ is a regular prime, we claim that $K_{2i}(\mathbb{Z}[\zeta_\ell])$ has no ℓ-torsion. (The case K_0 is tautological by 1.1, and the classical case K_2 is 1.4.) Note that the group $K_{2i-1}(\mathbb{Z}[\zeta_\ell]) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$ always has ℓ-torsion, because $w_i^{(\ell)}(F) \geq \ell$ for all i by 2.7(a). Setting $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, then by 6.5,

$$K_{2i}(\mathbb{Z}[\zeta_\ell]) \cong H^2_{\text{ét}}(R, \mathbb{Z}_\ell(i+1)) \oplus (\text{finite group without } \ell\text{-torsion}).$$

Since ℓ is regular, we have $\mathrm{Pic}(R)/\ell = 0$, and we saw in 1.4 that $\mathrm{Br}(R) = 0$ and $|S| = 1$. By 6.6, $H^2_{\text{ét}}(R, \mathbb{Z}_\ell(i+1)) = 0$ and the claim now follows.

We conclude with a comparison to the odd part of $\zeta_F(1-2k)$, generalizing the Birch-Tate Conjecture 2.14. If F is not totally real, $\zeta_F(s)$ has a pole of order r_2 at $s = 1-2k$. We need to invoke the following deep result of Wiles [Wi], which is often called the "Main Conjecture" of Iwasawa Theory.

Theorem 6.8 (Wiles). *Let F be a totally real number field. If ℓ is odd and $\mathcal{O}_S = \mathcal{O}_F[1/\ell]$ then for all even $i = 2k > 0$:*

$$\zeta_F(1-i) = \frac{|H^2_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i)|}{|H^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}_\ell(i)|} u_i,$$

where u_i is a rational number prime to ℓ.

The numerator and denominator on the right side are finite by 3.4. Lichtenbaum's conjecture follows, up to a power of 2, by setting $i = 2k$:

Theorem 6.9. *If F is totally real, then*

$$\zeta_F(1-2k) = (-1)^{kr_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|} \quad \textit{up to factors of 2.}$$

Proof. By the functional equation, the sign of $\zeta_F(1-2k)$ is $(-1)^{kr_1}$. It suffices to show that the left and right sides of 6.9 have the same power of each odd prime ℓ. The group $H^2_{\text{ét}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(i))$ is the ℓ-primary part of $K_{2i-2}(\mathcal{O}_F)$ by 6.2. The group $H^1_{\text{ét}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(i))$ on the bottom of 6.8 is $\mathbb{Z}/w_i^{(\ell)}(F)$ by 3.8.4, and this is isomorphic to the ℓ-primary subgroup of $K_{2i-1}(\mathcal{O}_F)$ by theorem 6.2. $\square$

§7. Real number fields at the prime 2

Let F be a real number field, i.e., F has $r_1 > 0$ embeddings into $\mathbb{R}$. The calculation of the algebraic K-theory of F at the prime 2 is somewhat different from the calculation at odd primes, for two reasons. One reason is that a real number field has infinite cohomological dimension, which complicates descent methods. A second reason is that the Galois group of a cyclotomic extension need not be cyclic, so that the e-invariant may not split (see 2.12). A final reason is that the groups $K_*(F;\mathbb{Z}/2)$ do not have a natural multiplication, because of the structure of the mod 2 Moore space $\mathbb{RP}^2$.

For the real numbers $\mathbb{R}$, the mod 2 motivic spectral sequence has $E_2^{p,q} = \mathbb{Z}/2$ for all p,q in the octant $q \le p \le 0$. In order to distinguish between the groups $E_2^{p,q}$, it is useful to label the nonzero elements of $H^0_{\text{ét}}(\mathbb{R},\mathbb{Z}/2(i))$ as β_i, writing 1 for β_0. Using the multiplicative pairing with (say) the spectral sequence ${}'E_2^{*,*}$ converging to $K_*(\mathbb{R};\mathbb{Z}/16)$, multiplication by the element $\eta \in {}'E_2^{0,-1}$ allows us to write the nonzero elements in the $-i$th column as $\eta^j\beta_i$. (See table 7.1.1 below)

From Suslin's calculation of $K_n(\mathbb{R})$ in [Su2], we know that the groups $K_n(\mathbb{R};\mathbb{Z}/2)$ are cyclic and 8-periodic (for $n \ge 0$) with orders $2,2,4,2,2,0,0,0$ (for $n = 0,1,...,7$).

Theorem 7.1. *In the spectral sequence converging to $K_*(\mathbb{R};\mathbb{Z}/2)$, all the d_2 differentials with nonzero source on the lines $p \equiv 1,2 \pmod 4$ are isomorphisms. Hence the spectral sequence degenerates at E_3. The only extensions are the nontrivial extensions $\mathbb{Z}/4$ in $K_{8a+2}(\mathbb{R};\mathbb{Z}/2)$.*

			1
		β_1	η
	β_2	$\eta\beta_1$	η^2
β_3	$\eta\beta_2$	$\eta^2\beta_1$	η^3
$\eta\beta_3$	$\eta^2\beta_2$	$\eta^3\beta_1$	η^4

The first 4 columns of E_2

			1
		β_1	η
	0	$\eta\beta_1$	η^2
0	0	$\eta^2\beta_1$	0
0	0	0	0

The first 4 columns of E_3

Table 7.1.1. The mod 2 spectral sequence for $\mathbb{R}$.

Proof. Recall from 0.6 that the mod 2 spectral sequence has periodicity isomorphisms $E_r^{p,q} \cong E_r^{p-4,q-4}$, $p \le 0$. Therefore it suffices to work with the columns $-3 \le p \le 0$.

Because $K_3(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$, the differential closest to the origin, from β_2 to η^3, must be nonzero. Since the pairing with ${}'E_2$ is multiplicative and $d_2(\eta) = 0$, we must have $d_2(\eta^j\beta_2) = \eta^{j+3}$ for all $j \ge 0$. Thus the column $p = -2$ of E_3 is zero, and every term in the column $p = 0$ of E_3 is zero except for $\{1,\eta,\eta^2\}$.

Similarly, we must have $d_2(\beta_3) = \eta^3\beta_1$ because $K_5(\mathbb{R};\mathbb{Z}/2) = 0$. By multiplicativity, this yields $d_2(\eta^j\beta_3) = \eta^{j+3}\beta_1$ for all $j \ge 0$. Thus the column $p = -3$ of E_3 is zero, and every term in the column $p = -1$ of E_3 is zero except for $\{\beta_1,\eta\beta_1,\eta^2\beta_1\}$. □

7.1.2. The analysis with coefficients $\mathbb{Z}/2^\infty$ is very similar, except that when $p > q$, $E_2^{p,q} = H^{p-q}_{\text{ét}}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: 0 for p even; $\mathbb{Z}/2$ for p odd. If p is odd, the coefficient map $\mathbb{Z}/2 \to \mathbb{Z}/2^\infty$ induces isomorphisms on the $E_2^{p,q}$ terms, so by 7.1 all the d_2 differentials with nonzero source in the columns $p \equiv 1 \pmod 4$ are isomorphisms. Again,

the spectral sequence converging to $K_*(\mathbb{R};\mathbb{Z}/2^\infty)$ degenerates at $E_3 = E_\infty$. The only extensions are the nontrivial extensions of $\mathbb{Z}/2^\infty$ by $\mathbb{Z}/2$ in $K_{8a+4}(\mathbb{R};\mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty$.

7.1.3. The analysis with 2-adic coefficients is very similar, except that (a) $H^0(\mathbb{R};\mathbb{Z}_2(i))$ is: $\mathbb{Z}_2$ for i even; 0 for i odd and (b) (for $p > q$) $E_2^{p,q} = H^{p-q}_{\text{ét}}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: $\mathbb{Z}/2$ for p even; 0 for p odd. All differentials with nonzero source in the column $p \equiv 2 \pmod 4$ are onto. Since there are no extensions to worry about, we omit the details.

In order to state the theorem 7.3 below for a ring $\mathcal{O}_S$ of integers in a number field F, we consider the natural maps (for $n > 0$) induced by the r_1 real embeddings of F,

$$\alpha^n_S(i)\colon H^n(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \to \bigoplus^{r_1} H^n(\mathbb{R};\mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i-n \text{ odd} \\ 0, & i-n \text{ even.} \end{cases} \tag{7.2.0}$$

This map is an isomorphism for all $n \geq 3$ by Tate-Poitou duality; by 3.8.3, it is also an isomorphism for $n = 2$ and $i \geq 2$. Write $\tilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for the kernel of $\alpha^1_S(i)$.

Lemma 7.2. *The map $H^1(F;\mathbb{Z}/2^\infty(i)) \xrightarrow{\alpha^1(i)} (\mathbb{Z}/2)^{r_1}$ is a split surjection for all even i. Hence $H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2)^{r_1} \oplus \tilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for sufficiently large S.*

Proof. By the strong approximation theorem for units of F, the left map vertical map is a split surjection in the diagram:

$$\begin{array}{ccccc} F^\times/F^{\times 2} & \xrightarrow{\cong} & H^1(F,\mathbb{Z}/2) & \to & H^1(F,\mathbb{Z}/2^\infty(i)) \\ {\scriptstyle\text{onto}}\downarrow{\scriptstyle\oplus\sigma} & & \downarrow & & \downarrow{\scriptstyle\alpha^1(i)} \\ (\mathbb{Z}/2)^{r_1} = \oplus\,\mathbb{R}^\times/\mathbb{R}^{\times 2} & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2) & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2^\infty(i)). \end{array}$$

Since $F^\times/F^{\times 2}$ is the direct limit (over S) of the groups $\mathcal{O}_S^\times/\mathcal{O}_S^{\times 2}$, we may replace F by $\mathcal{O}_S$ for sufficiently large S. □

We also write $A \rtimes B$ for an abelian group extension of B by A.

Theorem 7.3. *(*[RW, 6.9]*) Let F be a real number field, and let $R = \mathcal{O}_S$ be a ring of S-integers in F containing $\mathcal{O}_F[\frac{1}{2}]$. Then $\alpha^1_S(i)$ is onto when $i = 4k > 0$, and:*

$$K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty) \cong \begin{cases} \mathbb{Z}/w_{4k}(F) & \text{for } n = 8a, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+1)) & \text{for } n = 8a+1, \\ \mathbb{Z}/2 & \text{for } n = 8a+2, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+2)) & \text{for } n = 8a+3, \\ \mathbb{Z}/2w_{4k+2} \oplus (\mathbb{Z}/2)^{r_1-1} & \text{for } n = 8a+4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+3)) & \text{for } n = 8a+5, \\ 0 & \text{for } n = 8a+6, \\ \tilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+4)) & \text{for } n = 8a+7. \end{cases}$$

Proof. The morphism of spectral sequences (0.5), from that for $\mathcal{O}_S$ to the sum of r_1 copies of that for $\mathbb{R}$, is an isomorphism on $E_2^{p,q}$ except on the diagonal $p = q$ (where it

is an injection) and $p = q+1$ (where we must show it is a surjection). When $p \equiv +1 \pmod 4$, it follows from 7.1.2 that we may identify $d_2^{p,q}$ with α_S^{p-q}. Hence $d_2^{p,q}$ is an isomorphism if $p \geq 2+q$, and an injection if $p = q$. As in 7.1.2, the spectral sequence degenerates at E_3, yielding $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ as proclaimed, except for two points: (a) the extension of $\mathbb{Z}/w_{4a+2}$ by $\mathbb{Z}/2^{r_1}$ when $n = 8a+4$ is seen to be nontrivial by comparison with the extension for $\mathbb{R}$), and (b) when $n = 8a+6$ it only shows that $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ is the cokernel of $\alpha_S^1(4a+4)$.

To resolve (b) we must show that $\alpha_S^1(4a+4)$ is onto when $a > 0$. Set $n = 8a+6$. Since $K_n(\mathcal{O}_S)$ is finite, $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ must equal the 2-primary subgroup of $K_{n-1}(\mathcal{O}_S)$, which is independent of S by 1.6. But for sufficiently large S, the map $\alpha^1(4a+4)$ is a surjection by 7.2, and hence $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty) = 0$. $\square$

Proof of Theorem 0.1. Let $n > 0$ be odd. By 1.5 and 1.6, it suffices to determine the torsion subgroup of $K_n(\mathcal{O}_S) = K_n(F)$. Since $K_{n+1}(\mathcal{O}_S)$ is finite, it follows that $K_{n+1}(\mathcal{O}_S; \mathbb{Z}/\ell^\infty)$ is the ℓ-primary subgroup of $K_n(\mathcal{O}_S)$. By 6.5, we may assume F has a real embedding. By 6.2, we need only worry about the 2-primary torsion, which we can read off from 7.3, recalling from 2.8(b) that $w_i^{(2)}(F) = 2$ for odd i. $\square$

To proceed further, we need to introduce the narrow Picard group and the signature defect of the ring $\mathcal{O}_S$.

Narrow Picard group 7.4. Each real embedding $\sigma_i : F \to \mathbb{R}$ determines a map $F^\times \to \mathbb{R}^\times \to \mathbb{Z}/2$, detecting the sign of units of F under that embedding. The sum of these maps is the *sign map* $\sigma\colon F^\times \to (\mathbb{Z}/2)^{r_1}$. The approximation theorem for F implies that σ is surjective. The group $F_+^\times$ of *totally positive units* in F is defined to be the kernel of σ.

Now let $R = \mathcal{O}_S$ be a ring of integers in F. The kernel of $\sigma|_R : R^\times \to F^\times \to (\mathbb{Z}/2)^{r_1}$ is the subgroup $R_+^\times$ of totally positive units in R. Since the sign map $\sigma|_R$ factors through $F^\times/2 = H^1(F, \mathbb{Z}/2)$, it also factors through $\alpha^1 : H^1(R, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$. The *signature defect* $j(R)$ of R is defined to be the dimension of the cokernel of α^1; $0 \leq j(R) < r_1$ because $\sigma(-1) \neq 0$. Note that $j(F) = 0$, and that $j(R) \leq j(\mathcal{O}_F)$.

By definition, the *narrow Picard group* $\mathrm{Pic}_+(R)$ is the cokernel of the the restricted divisor map $F_+^\times \to \bigoplus_{\wp \notin S} \mathbb{Z}$. (See [Co, 5.2.7]. This definition is due to Weber; $\mathrm{Pic}_+(\mathcal{O}_S)$ is also called the *ray class group* Cl_F^S; see [Neu, VI.1].) The kernel of the restricted divisor map is clearly $R_+^\times$, and it is easy to see from this that there is an exact sequence

$$0 \to R_+^\times \to R^\times \xrightarrow{\sigma} (\mathbb{Z}/2)^{r_1} \to \mathrm{Pic}_+(R) \to \mathrm{Pic}(R) \to 0.$$

A diagram chase (performed in [RW, 7.6]) shows that there is an exact sequence

$$(7.4.1) \quad 0 \to \tilde{H}^1(R; \mathbb{Z}/2) \to H^1(R; \mathbb{Z}/2) \xrightarrow{\alpha^1} (\mathbb{Z}/2)^{r_1} \to \mathrm{Pic}_+(R)/2 \to \mathrm{Pic}(R)/2 \to 0.$$

($\tilde{H}^1(R; \mathbb{Z}/2)$ is defined as the kernel of α^1.) Thus the signature defect $j(R)$ is also the dimension of the kernel of $\mathrm{Pic}_+(R)/2 \to \mathrm{Pic}(R)/2$. If we let t and u denote the dimensions of $\mathrm{Pic}(R)/2$ and $\mathrm{Pic}_+(R)/2$, respectively, then this means that $u = t + j(R)$.

If s denotes the number of finite places of $R = \mathcal{O}_S$, then $\dim H^1(R;\mathbb{Z}/2) = r_1 + r_2 + s + t$ and $\dim H^2(R;\mathbb{Z}/2) = r_1 + s + t - 1$. This follows from (1.2.1) and (1.3.1), using Kummer theory. As in (7.2.0) and (7.4.1), define $\tilde{H}^n(R;\mathbb{Z}/2)$ to be the kernel of $\alpha^n : H^n(R;\mathbb{Z}/2) \to H^n(\mathbb{R};\mathbb{Z}/2)^{r_1} \cong (\mathbb{Z}/2)^{r_1}$.

Lemma 7.4.2. *Suppose that* $\frac{1}{2} \in R$. *Then* $\dim \tilde{H}^1(R,\mathbb{Z}/2) = r_2 + s + u$. *Moreover, the map* $\alpha^2 : H^2(R,\mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$ *is onto, and* $\dim \tilde{H}^2(R,\mathbb{Z}/2) = t + s - 1$.

Proof. The first assertion is immediate from (7.4.1). Since $H^2(R;\mathbb{Z}/2^\infty(4)) \cong (\mathbb{Z}/2)^{r_1}$ by (3.8.3), the coefficient sequence for $\mathbb{Z}/2 \subset \mathbb{Z}/2^\infty(4)$ shows that $H^2(R;\mathbb{Z}/2) \to H^2(R;\mathbb{Z}/2^\infty(4))$ is onto. The final two assertions follow. □

Theorem 7.5. *Let F be a real number field, and $\mathcal{O}_S$ a ring of integers containing $\frac{1}{2}$. If $j = j(\mathcal{O}_S)$ is the signature defect, then the mod 2 algebraic K-groups of $\mathcal{O}_S$ are given (up to extensions) for $n > 0$ as follows:*

$$K_n(\mathcal{O}_S;\mathbb{Z}/2) \cong \begin{cases} \tilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2) \oplus \mathbb{Z}/2 & \text{for } n = 8a, \\ H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+1, \\ H^2(\mathcal{O}_S;\mathbb{Z}/2) \rtimes \mathbb{Z}/2 & \text{for } n = 8a+2, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+3, \\ (\mathbb{Z}/2)^{j} \rtimes H^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes \tilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+5, \\ (\mathbb{Z}/2)^{j} \oplus \tilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+6, \\ \tilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n = 8a+7. \end{cases}$$

			1
		β_1	H^1
	0	H^1	H^2
0	$\tilde{H}^1$	H^2	$(\mathbb{Z}/2)^{r_1-1}$
$\tilde{H}^1$	$\tilde{H}^2$	$(\mathbb{Z}/2)^{r_1-1}$	$(\mathbb{Z}/2)^{j}$
$\tilde{H}^2$	0	$(\mathbb{Z}/2)^{j}$	0
0	0	0	0

The first 4 columns ($-3 \le p \le 0$) of $E_3 = E_\infty$

Table 7.5.1. The mod 2 spectral sequence for $\mathcal{O}_S$.

Proof. (Cf. [RW, 7.8].) As in the proof of Theorem 7.3, we compare the spectral sequence for $R = \mathcal{O}_S$ with the sum of r_1 copies of the spectral sequence for $\mathbb{R}$. For $n \ge 3$ we have $H^n(R;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$. It is not hard to see that we may identify the differentials $d_2 : H^n(R,\mathbb{Z}/2) \to H^{n+3}(R,\mathbb{Z}/2)$ with the maps α^n. Since these maps are described in 7.4.2, we see from 0.6 that that the columns $p \le 0$ of E_3 are 4-periodic, and all nonzero entries are described by Figure 7.5.1. (By (0.5), there is only one nonzero entry for $p > 0$, $E_3^{+1,-1} = \mathrm{Pic}(R)/2$, and it is only important for $n = 0$.) By inspection, $E_3 = E_\infty$,

yielding the desired description of the groups $K_n(R, \mathbb{Z}/2)$ in terms of extensions. We omit the proof that the extensions split if $n \equiv 0, 6 \pmod 8$. $\square$

The case $F = \mathbb{Q}$ has historical importance, because of its connection with the image of J (see 2.12 or [Q5]) and classical number theory. The following result was first established in [We2]; the groups are not truly periodic only because the order of $K_{8a-1}(\mathbb{Z})$ depends upon a.

Corollary 7.6. *For $n \geq 0$, the two-primary subgroups of $K_n(\mathbb{Z})$ and $K_2(\mathbb{Z}[1/2])$ are essentially periodic, of period eight, and are given by the following table. (When $n \equiv 7 \pmod 8$, we set $a = (n+1)/8$.)*

$n \pmod 8$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})\{2\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/16a$	0

In particular, $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ have odd order for all $n \equiv 4, 6, 8 \pmod 8$, and the finite group $K_{8a+2}(\mathbb{Z})$ is the sum of $\mathbb{Z}/2$ and a finite group of odd order. We will say more about the odd torsion in the next section.

Proof. When n is odd, this is theorem 0.1; $w_{4a}^{(2)}$ is the 2-primary part of $16a$ by 2.8(c). Since $s = 1$ and $t = u = 0$, we see from 7.4.2 that $\dim \tilde{H}^1(\mathbb{Z}[1/2]; \mathbb{Z}/2) = 1$ and that $\tilde{H}^2(\mathbb{Z}[1/2]; \mathbb{Z}/2) = 0$. By 7.5, the groups $K_n(\mathbb{Z}[1/2]; \mathbb{Z}/2)$ are periodic of orders 2, 4, 4, 4, 2, 2, 1, 2 for $n \equiv 0, 1, ..., 7$ respectively. The groups $K_n(\mathbb{Z}[1/2])$ for n odd, given in 0.1, together with the $\mathbb{Z}/2$ summand in $K_{8a+2}(\mathbb{Z})$ provided by topology (see 2.12), account for all of $K_n(\mathbb{Z}[1/2]; \mathbb{Z}/2)$, and hence must contain all of the 2-primary torsion in $K_n(\mathbb{Z}[1/2])$. $\square$

Recall that the 2-rank of an abelian group A is the dimension of $\mathrm{Hom}(\mathbb{Z}/2, A)$. We have already seen (in either theorem 0.1 or 7.3) that for $n \equiv 1, 3, 5, 7 \pmod 8$ the 2-ranks of $K_n(\mathcal{O}_S)$ are: 1, r_1, 0 and 1, respectively.

Corollary 7.7. *For $n \equiv 2, 4, 6, 8 \pmod 8$, $n > 0$, the respective 2-ranks of the finite groups $K_n(\mathcal{O}_S)$ are: $r_1 + s + t - 1$, $j + s + t - 1$, $j + s + t - 1$ and $s + t - 1$.*

Proof. (Cf. [RW, 0.7].) Since $K_n(R; \mathbb{Z}/2)$ is an extension of $\mathrm{Hom}(\mathbb{Z}/2, K_{n-1}R)$ by $K_n(R)/2$, and the dimensions of the odd groups are known, we can read this off from the list given in theorem 7.5. $\square$

Example 7.7.1. Consider $F = \mathbb{Q}(\sqrt{p})$, where p is prime. When $p \equiv 1 \pmod 8$, it is well known that $t = j = 0$ but $s = 2$. It follows that $K_{8a+2}(\mathcal{O}_F)$ has 2-rank 3, while the two-primary summand of $K_n(\mathcal{O}_F)$ is nonzero and cyclic when $n \equiv 4, 6, 8 \pmod 8$.

When $p \equiv 7 \pmod 8$, we have $j = 1$ for both $\mathcal{O}_F$ and $R = \mathcal{O}_F[1/2]$. Since $r_1 = 2$ and $s = 1$, the 2-ranks of the finite groups $K_n(R)$ are: $t + 2$, $t + 1$, $t + 1$ and t for $n \equiv 2, 4, 6, 8 \pmod 8$ by 7.7. For example, if $t = 0$ ($\mathrm{Pic}(R)/2 = 0$) then $K_n(R)$ has odd order for $n \equiv 8 \pmod 8$, but the 2-primary summand of $K_n(R)$ is $(\mathbb{Z}/2)^2$ when $n \equiv 2$ and is cyclic when $n \equiv 4, 6$.

Example 7.7.2. (2–regular fields) A number field F is said to be *2–regular* if there is only one prime over 2 and the narrow Picard group $\mathrm{Pic}_+(\mathcal{O}_F[\frac{1}{2}])$ is odd (*i.e.*, $t = u = 0$ and $s = 1$). In this case, we see from 7.7 that $K_{8a+2}(\mathcal{O}_F)$ is the sum of $(\mathbb{Z}/2)^{r_1}$ and a finite odd group, while $K_n(\mathcal{O}_F)$ has odd order for all $n \equiv 4, 6, 8 \pmod 8$ ($n > 0$). In particular, the map $K_4^M(F) \to K_4(F)$ must be zero, since it factors through the odd order group $K_4(\mathcal{O}_F)$, and $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Browkin and Schinzel [BS] and Rognes and Østvær [RØ] have studied this case. For example, when $F = \mathbb{Q}(\sqrt{m})$ and $m > 0$ ($r_1 = 2$), the field F is 2-regular exactly when $m = 2$, or $m = p$ or $m = 2p$ with $p \equiv 3, 5 \pmod 8$ prime. (See [BS].)

A useful example is $F = \mathbb{Q}(\sqrt{2})$. Note that the Steinberg symbols $\{-1, -1, -1, -1\}$ and $\{-1, -1, -1, 1 + \sqrt{2}\}$ generating $K_4^M(F) \cong (\mathbb{Z}/2)^2$ must both vanish in $K_4(\mathbb{Z}[\sqrt{2}])$, which we have seen has odd order. This is the case $j = \rho = 0$ of the following result.

Corollary 7.8. *Let F be a real number field. Then the rank ρ of the image of $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(F)$ satisfies $j(\mathcal{O}_F[1/2]) \le \rho \le r_1 - 1$. The image $(\mathbb{Z}/2)^\rho$ lies in the subgroup $K_4(\mathcal{O}_F)$ of $K_4(F)$, and its image in $K_4(\mathcal{O}_S)/2$ has rank $j(\mathcal{O}_S)$ for all $\mathcal{O}_S$ containing $1/2$. In particular, the image $(\mathbb{Z}/2)^\rho$ lies in $2 \cdot K_4(F)$.*

Proof. By 1.8, we have $\rho < r_1 = \mathrm{rank}\ K_4^M(F)$. The assertion that $K_4^M(F) \to K_4(F)$ factors through $K_4(\mathcal{O}_F)$ follows from 1.7, by multiplying $K_3^M(F)$ and $K_3(\mathcal{O}_F) \cong K_3(F)$ by $[-1] \in K_1(\mathbb{Z})$. It is known [SF, 15.5] that the edge map $H^n(F, \mathbb{Z}(n)) \to K_n(F)$ in the motivic spectral sequence agree with the usual map $K_n^M(F) \to K_n(F)$. By Voevodsky's theorem, $K_n^M(F)/2^\nu \cong H^n(F, \mathbb{Z}(n))/2^\nu \cong H^n(F, \mathbb{Z}/2^\nu(n))$. For $n = 4$, the image of the edge map from $H^4(\mathcal{O}_S, \mathbb{Z}/2^\nu(4)) \cong H^4(F, \mathbb{Z}/2^\nu(4)) \to K_4(\mathcal{O}_S; \mathbb{Z}/2)$ has rank j by table 7.5.1; this implies the assertion that the image in $K_4(\mathcal{O}_S)/2 \subset K_4(\mathcal{O}_S; \mathbb{Z}/2)$ has rank $j(\mathcal{O}_S)$. Finally, taking $\mathcal{O}_S = \mathcal{O}_F[1/2]$ yields the inequality $j(\mathcal{O}_S) \le \rho$. $\square$

Example 7.8.1. ($\rho = 1$) Consider $F = \mathbb{Q}(\sqrt{7})$, $\mathcal{O}_F = \mathbb{Z}[\sqrt{7}]$ and $R = \mathcal{O}_F[1/2]$; here $s = 1$, $t = 0$ and $j(R) = \rho = 1$ (the fundamental unit $u = 8 + 3\sqrt{7}$ is totally positive). Hence the image of $K_4^M(F) \cong (\mathbb{Z}/2)^2$ in $K_4(\mathbb{Z}[\sqrt{7}])$ is $\mathbb{Z}/2$ on the symbol $\sigma = \{-1, -1, -1, \sqrt{7}\}$, and this is all of the 2-primary torsion in $K_4(\mathbb{Z}[\sqrt{7}])$ by 7.7.

On the other hand, $\mathcal{O}_S = \mathbb{Z}[\sqrt{7}, 1/7]$ still has $\rho = 1$, but now $j = 0$, and the 2-rank of $K_4(\mathcal{O}_S)$ is still one by 7.7. Hence the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/48 \to 0$ of 1.6 cannot be split, implying that the 2-primary subgroup of $K_4(\mathcal{O}_S)$ must then be $\mathbb{Z}/32$.

In fact, the nonzero element σ is divisible in $K_4(F)$. This follows from the fact that if $p \equiv 3 \pmod{28}$ then there is an irreducible $q = a + b\sqrt{7}$ whose norm is $-p = q\bar{q}$. Hence $R' = \mathbb{Z}[\sqrt{7}, 1/2q]$ has $j(R') = 0$ but $\rho = 1$, and the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/(p^2 - 1) \to 0$ of 1.6 is not split. If in addition $p \equiv -1 \pmod{2^\nu}$ — there are infinitely many such p for each ν — then there is an element v of $K_4(R')$ such that $2^{\nu+1}v = \sigma$ See [We3] for details.

Question 7.8.2. Can ρ be less than the minimum of $r_1 - 1$ and $j + s + t - 1$?

As in (7.2.0), when i is even we define $\tilde{H}^2(R; \mathbb{Z}_2(i))$ to be the kernel of $\alpha^2(i) : H^2(R; \mathbb{Z}_2(i)) \to H^2(\mathbb{R}; \mathbb{Z}_2(i))^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. By 7.4.2, $\tilde{H}^2(R; \mathbb{Z}_2(i))$ has 2-rank $s + t - 1$.

Theorem 7.9. *([RW, 0.6]) Let F be a number field with at least one real embedding, and let $R = \mathcal{O}_S$ denote a ring of integers in F containing $1/2$. Let j be the signature defect of R, and write w_i for $w_i^{(2)}(F)$.*

Then there is an integer ρ, $j \le \rho < r_1$, such that, for all $n \ge 2$, the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of $K_n(\mathcal{O}_S)$ is isomorphic to:

$$K_n(\mathcal{O}_S)\{2\} \cong \begin{cases} H^2_{\text{ét}}(R;\mathbb{Z}_2(4a+1)) & \text{for } n = 8a, \\ \mathbb{Z}/2 & \text{for } n = 8a+1, \\ H^2_{\text{ét}}(R;\mathbb{Z}_2(4a+2)) & \text{for } n = 8a+2, \\ (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}/2w_{4a+2} & \text{for } n = 8a+3, \\ (\mathbb{Z}/2)^{\rho} \rtimes H^2_{\text{ét}}(R;\mathbb{Z}_2(4a+3)) & \text{for } n = 8a+4, \\ 0 & \text{for } n = 8a+5, \\ \tilde{H}^2_{\text{ét}}(R;\mathbb{Z}_2(4a+4)) & \text{for } n = 8a+6, \\ \mathbb{Z}/w_{4a+4} & \text{for } n = 8a+7. \end{cases}$$

Proof. When $n = 2i-1$ is odd, this is theorem 0.1, since $w_i^{(2)}(F) = 2$ when $n \equiv 1 \pmod 4$ by 2.8(b). When $n = 2$ it is 1.3. To determine the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of the finite group $K_{2i+2}(\mathcal{O}_S)$ when $n = 2i+2$, we use the universal coefficient sequence

$$0 \to (\mathbb{Z}/2^\infty)^r \to K_{2i+3}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \to K_{2i+2}(\mathcal{O}_S)\{2\} \to 0,$$

where r is the rank of $K_{2i+3}(\mathcal{O}_S)$ and is given by 1.5 ($r = r_1 + r_2$ or r_2). To compare this with theorem 7.3, we note that $H^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i))$ is the direct sum of $(\mathbb{Z}/2^\infty)^r$ and a finite group, which must be $H^2(\mathcal{O}_S, \mathbb{Z}_2(i))$ by universal coefficients; see [RW, 2.4(b)]. Since $\alpha^1_S(i) : H^1(R;\mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ must vanish on the divisible group $(\mathbb{Z}/2^\infty)^r$, it induces the natural map $\alpha^2_S(i)\colon H^2_{\text{ét}}(\mathcal{O}_S;\mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ and

$$\tilde{H}^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2^\infty)^r \oplus \tilde{H}^2(\mathcal{O}_S, \mathbb{Z}_2(i)).$$

This proves all of the theorem, except for the description of $K_n(\mathcal{O}_S)$, $n = 8a+4$. By mod 2 periodicity 0.6, the integer ρ of 7.8 equals the rank of the image of $H^4(\mathcal{O}_S, \mathbb{Z}/2(4)) \cong H^4(\mathcal{O}_S, \mathbb{Z}/2(4k+4)) \cong (\mathbb{Z}/2)^{r_1}$ in $\mathrm{Hom}(\mathbb{Z}/2, K_n(\mathcal{O}_S))$, considered as a quotient of $K_{n+1}(\mathcal{O}_S;\mathbb{Z}/2)$. □

We can combine the 2-primary information in 7.9 with the odd torsion information in 6.2 and 6.9 to relate the orders of K-groups to the orders of étale cohomology groups. Up to a factor of 2^{r_1}, they were conjectured by Lichtenbaum in [Li2]. Let $|A|$ denote the order of a finite abelian group A.

Theorem 7.10. *Let F be a totally real number field, with r_1 real embeddings, and let $\mathcal{O}_S$ be a ring of integers in F. Then for all even $i > 0$*

$$2^{r_1} \cdot \frac{|K_{2i-2}(\mathcal{O}_S)|}{|K_{2i-1}(\mathcal{O}_S)|} = \frac{\prod_\ell |H^2_{\text{ét}}(\mathcal{O}_S[1/\ell];\mathbb{Z}_2(i))|}{\prod_\ell |H^1_{\text{ét}}(\mathcal{O}_S[1/\ell];\mathbb{Z}_2(i))|}.$$

Proof. (Cf. proof of 6.9.) Since $2i-1 \equiv 3 \pmod 4$, all groups involved are finite (see 1.5, 3.8 and 3.8.4.) Write $h^{n,i}(\ell)$ for the order of $H^n_{\text{ét}}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))$. By 3.8.4, $h^{1,i}(\ell) = w_i^{(\ell)}(F)$. By 0.1, the ℓ-primary subgroup of $K_{2i-1}(\mathcal{O}_S)$ has order $h^{1,i}(\ell)$ for all odd ℓ and all even $i > 0$, and also for $\ell = 2$ with the exception that when $2i-1 \equiv 3 \pmod 8$ then the order is $2^{r_1} h^{1,i}(2)$.

By 6.2 and 7.9, the ℓ-primary subgroup of $K_{2i-2}(\mathcal{O}_S)$ has order $h^{2,i}(\ell)$ for all ℓ, except when $\ell = 2$ and $2i-2 \equiv 6 \pmod 8$ when it is $h^{1,i}(2)/2^{r_1}$. Combining these cases yields the formula asserted by the theorem. □

Corollary 7.11. *For $R = \mathbb{Z}$, the formula conjectured by Lichtenbaum in* [Li2] *holds up to exactly one factor of 2. That is, for $k \geq 1$,*

$$\frac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|} = \frac{B_k}{4k} = \frac{(-1)^k}{2}\zeta(1-2k).$$

Moreover, if c_k denotes the numerator of $\frac{B_k}{4k}$, then

$$|K_{4k-2}(\mathbb{Z})| = \begin{cases} c_k, & k \text{ even} \\ 2\ c_k, & k \text{ odd.} \end{cases}$$

Proof. The equality $B_k/4k = (-1)^k\zeta(1-2k)/2$ comes from 2.10.1. By 6.9, the formula holds up to a factor of 2. By 2.11, the two-primary part of $B_k/4k$ is $1/w_{2k}^{(2)}$. By 2.8(c), this is also the two-primary part of $1/8k$. By 7.6, the two-primary part of the left-hand side of 7.11 is $2/16$ when k is odd, and the two-primary part of $1/8k$ when $k = 2a$ is even. □

Examples 7.12. ($K_{4k-2}(\mathbb{Z})$) The group $K_{4k-2}(\mathbb{Z})$ is cyclic of order c_k or $2c_k$ for all $k \leq 5000$. For small k we need only consult 2.10 to see that the groups $K_2(\mathbb{Z})$, $K_{10}(\mathbb{Z})$, $K_{18}(\mathbb{Z})$ and $K_{26}(\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2$. We also have $K_6(\mathbb{Z}) = K_{14}(\mathbb{Z}) = 0$. (The calculation of $K_6(\mathbb{Z})$ up to 3-torsion was given in [EGS].) However, $c_6 = 691$, $c_8 = 3617$, $c_9 = 43867$ and $c_{13} = 657931$ are all prime, so we have $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691$, $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$, $K_{34}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/43867$ and $K_{50} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/657931$.

The next hundred values of c_k are squarefree: $c_{10} = 283 \cdot 617$, $c_{11} = 131 \cdot 593$, $c_{12} = 103 \cdot 2294797$, $c_{14} = 9349 \cdot 362903$ and $c_{15} = 1721 \cdot 1001259881$ are all products of two primes, while $c_{16} = 37 \cdot 683 \cdot 305065927$ is a product of 3 primes. Hence $K_{38}(\mathbb{Z}) = \mathbb{Z}/c_{10}$, $K_{42}(\mathbb{Z}) = \mathbb{Z}/2c_{11}$, $K_{46} = \mathbb{Z}/c_{12}$, $K_{54}(\mathbb{Z}) = \mathbb{Z}/c_{14}$, $K_{58}(\mathbb{Z}) = \mathbb{Z}/2c_{15}$ and $K_{62}(\mathbb{Z}) = \mathbb{Z}/c_{16} = \mathbb{Z}/37 \oplus \mathbb{Z}/683 \oplus \mathbb{Z}/305065927$.

Thus the first occurrence of the smallest irregular prime (37) is in $K_{62}(\mathbb{Z})$; it also appears as a $\mathbb{Z}/37$ summand in $K_{134}(\mathbb{Z})$, $K_{206}(\mathbb{Z})$, ... , $K_{494}(\mathbb{Z})$. In fact, there is 37-torsion in every group $K_{72a+62}(\mathbb{Z})$ (see 8.6 below).

For $k < 5000$, only seven of the c_k are not square-free; see [OEIS, A090943]. The numerator c_k is divisible by ℓ^2 only for the following pairs (k, ℓ): $(114, 103)$, $(142, 37)$, $(457, 59)$, $(717, 271)$, $(1646, 67)$ and $(2884, 101)$. However, $K_{4k-2}(\mathbb{Z})$ is still cyclic with one $\mathbb{Z}/\ell^2$ summand in these cases. To see this, we note that $\mathrm{Pic}(R)/\ell \cong \mathbb{Z}/\ell$ for these ℓ, where $R = \mathbb{Z}[\zeta_\ell]$. Hence $K_{4k-2}(R)/\ell \cong H^2(R, \mathbb{Z}_\ell(2k))/\ell \cong H^2(R, \mathbb{Z}/\ell(2k)) \cong \mathrm{Pic}(R) \cong \mathbb{Z}/\ell$. The usual transfer argument now shows that $K_{4k-2}(\mathbb{Z})/\ell$ is either zero or $\mathbb{Z}/\ell$ for all k.

§8. The odd torsion in $K_*(\mathbb{Z})$

We now turn to the ℓ-primary torsion in the K-theory of $\mathbb{Z}$, where ℓ is an odd prime. By 2.11 and 6.2, the odd-indexed groups $K_{2i-1}(\mathbb{Z})$ have ℓ-torsion exactly when $i \equiv 0 \pmod{\ell-1}$. Thus we may restrict attention to the groups $K_{2i}(\mathbb{Z})$, whose ℓ-primary subgroups are $H^2_{\text{ét}}(\mathbb{Z}[1/\ell]; \mathbb{Z}_\ell(i+1))$ by 6.2.

Our method is tonsider the cyclotomic extension $\mathbb{Z}[\zeta]$ of $\mathbb{Z}$, $\zeta = e^{2\pi i/\ell}$. Because the Galois group $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is cyclic of order $\ell-1$, prime to ℓ, the usual transfer argument shows that $K_*(\mathbb{Z}) \to K_*(\mathbb{Z}[\zeta])$ identifies $K_n(\mathbb{Z}) \otimes \mathbb{Z}_\ell$ with $K_n(\mathbb{Z}[\zeta]))^G \otimes \mathbb{Z}_\ell$ for all n. Because $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/\ell])$ have the same ℓ-torsion (by the localization sequence), it suffices to work with $\mathbb{Z}[1/\ell]$.

Proposition 8.1. *When ℓ is an odd regular prime there is no ℓ-torsion in $K_{2i}(\mathbb{Z})$.*

Proof. Since ℓ is regular, we saw in example 6.7 that the finite group $K_{2i}(\mathbb{Z}[\zeta])$ has no ℓ-torsion. Hence the same is true for its G-invariant subgroup, $K_{2i}(\mathbb{Z})$. □

It follows from this and 2.11 that $K_{2i}(\mathbb{Z}; \mathbb{Z}/\ell)$ contains only the Bockstein representatives of the Harris-Segal summands in $K_{2i-1}(\mathbb{Z})$, and this only when $2i \equiv 0 \pmod{2\ell-2}$.

We can also describe the algebra structure of $K_*(\mathbb{Z}; \mathbb{Z}/\ell)$ using the action of the cyclic group $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ on the ring $K_*(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$. For simplicity, let us assume that ℓ is a regular prime. It is useful to set $R = \mathbb{Z}[\zeta, 1/\ell]$ and recall from 8.3 that $K_* = K_*(R; \mathbb{Z}/\ell)$ is a free graded $\mathbb{Z}/\ell[\beta]$-module on the $\frac{\ell+1}{2}$ generators of $R^\times/\ell \in K_1(R; \mathbb{Z}/\ell)$, together with $1 \in K_0(R; \mathbb{Z}/\ell)$.

By Maschke's theorem, $\mathbb{Z}/\ell[G] \cong \prod_{i=0}^{\ell-2} \mathbb{Z}/\ell$ is a simple ring; every $\mathbb{Z}/\ell[G]$-module has a unique decomposition as a sum of irreducible modules. Since μ_ℓ is an irreducible G-module, it is easy to see that the irreducible G-modules are $\mu_\ell^{\otimes i}$, $i = 0, 1, ..., \ell-2$. The "trivial" G-module is $\mu_\ell^{\otimes \ell-1} = \mu_\ell^{\otimes 0} = \mathbb{Z}/\ell$. By convention, $\mu_\ell^{\otimes -i} = \mu_\ell^{\otimes \ell-1-i}$.

For example, the G-module $\langle \beta^i \rangle$ of $K_{2i}(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$ generated by β^i is isomorphic to $\mu_\ell^{\otimes i}$. It is a trivial G-module only when $(\ell-1)|i$.

If A is any $\mathbb{Z}/\ell[G]$-module, it is traditional to decompose $A = \oplus A^{[i]}$, where $A^{[i]}$ denotes the sum of all G-submodules isomorphic to $\mu_\ell^{\otimes i}$.

Example 8.2. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$. It is known that the torsionfree part $R^\times/\mu_\ell \cong \mathbb{Z}^{\frac{\ell-1}{2}}$ of the units of R is isomorphic as a G-module to $\mathbb{Z}[G] \otimes_{\mathbb{Z}[c]} \mathbb{Z}$, where c is complex conjugation. (This is sometimes included as part of Dirichlet's theorem on units.) It follows that as a G-module,

$$H^1_{\text{ét}}(R, \mu_\ell) = R^\times/R^{\times \ell} \cong \mu_\ell \oplus (\mathbb{Z}/\ell) \oplus \mu_\ell^{\otimes 2} \oplus \cdots \oplus \mu_\ell^{\otimes \ell-3}.$$

The root of unity ζ generates the G-submodule μ_ℓ, and the class of the unit ℓ of R generates the trivial submodule of $R^\times/R^{\times \ell}$.

Tensoring with $\mu_\ell^{\otimes i-1}$ yields the G-module decomposition of $R^\times \otimes \mu_\ell^{\otimes i}$. If ℓ is regular this is $K_{2i-1}(R; \mathbb{Z}/\ell) \cong H^1_{\text{ét}}(R, \mu_\ell^{\otimes i})$ by 6.4. If i is even, exactly one term is $\mathbb{Z}/\ell$; if i is odd, $\mathbb{Z}/\ell$ occurs only when $i \equiv 0 \pmod{\ell-1}$.

Notation 8.2.1. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, For $i = 0, ..., \frac{\ell-3}{2}$, pick a generator x_i of the G-submodule of $R^\times/R^{\times\ell}$ isomorphic to $\mu_\ell^{\otimes -2i}$. The indexing is set up so that $y_i = \beta^{2i}x_i$ is a G-invariant element of $K_{4i+1}(R;\mathbb{Z}/\ell) \cong H^1_{\text{ét}}(R, \mu_\ell^{\otimes 2i+1})$. We may arrange that $x_0 = y_0$ is the unit $[\ell]$ in $K_1(R;\mathbb{Z}/\ell)$.

The elements $\beta^{\ell-1}$ of $H^0_{\text{ét}}(R, \mu_\ell^{\otimes \ell-1})$ and $v = \beta^{\ell-2}[\zeta]$ of $H^1_{\text{ét}}(R, \mu_\ell^{\otimes \ell-1})$ are also G-invariant. By abuse of notation, we shall also write $\beta^{\ell-1}$ and v, respectively, for the corresponding elements of $K_{2\ell-2}(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ and $K_{2\ell-3}(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$.

Theorem 8.3. *If ℓ is an odd regular prime then $K_* = K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is a free graded module over the polynomial ring $\mathbb{Z}/\ell[\beta^{\ell-1}]$. It has $(\ell+3)/2$ generators: $1 \in K_0$, $v \in K_{2\ell-3}$, and $y_i \in K_{4i+1}$ $(i = 0, ..., \frac{\ell-3}{2})$.*

Similarly, $K_*(\mathbb{Z};\mathbb{Z}/\ell)$ is a free graded module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$; a generating set is obtained from the generators of K_* by replacing y_0 by $y_0\beta^{\ell-1}$.

The submodule generated by v and $\beta^{\ell-1}$ comes from the Harris-Segal summands of $K_{4i-1}(\mathbb{Z})$. The submodule generated by the y's comes from the $\mathbb{Z}$ summands in $K_{4i+1}(\mathbb{Z})$.

Proof. $K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is the G-invariant subalgebra of $K_*(R;\mathbb{Z}/\ell)$. Given 8.2, it is not very hard to check that this is just the subalgebra described in the theorem. □

Examples 8.3.1. When $\ell = 3$, the groups $K_* = K_*(\mathbb{Z}[1/3];\mathbb{Z}/3)$ are 4-periodic of ranks $1, 1, 0, 1$, generated by an appropriate power of β^2 times one of $\{1, [3], v\}$.

When $\ell = 5$, the groups $K_* = K_*(\mathbb{Z}[1/5];\mathbb{Z}/5)$ are 8-periodic, with respective ranks $1, 1, 0, 0, 0, 1, 0, 1$ $(* = 0, ..., 7)$, generated by an appropriate power of β^4 times one of $\{1, [5], y_1, v\}$.

Now suppose that ℓ is an irregular prime, so that $\mathrm{Pic}(R)$ has ℓ-torsion for $R = \mathbb{Z}[\zeta, 1/\ell]$. Then $H^1_{\text{ét}}(R, \mu_\ell)$ is $R^\times/\ell \oplus {}_\ell\,\mathrm{Pic}(R)$ and $H^2_{\text{ét}}(R, \mu_\ell) \cong \mathrm{Pic}(R)/\ell$ by Kummer theory. This yields $K_*(R;\mathbb{Z}/\ell)$ by 6.4.

Example 8.4. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$ and $P = \mathrm{Pic}(R)/\ell$. If ℓ is regular then $P = 0$ by definition 1.1. When ℓ is irregular, the G-module structure of P is not fully understood; see Vandiver's conjecture 8.5 below. However, the following arguments show that $P^{[i]} = 0$, *i.e.*, P contains no summands isomorphic to $\mu_\ell^{\otimes i}$, for $i = 0, -1, -2, -3$.

The usual transfer argument shows that $P^G \cong \mathrm{Pic}(\mathbb{Z}[1/\ell])/\ell = 0$. Hence P contains no summands isomorphic to $\mathbb{Z}/\ell$. By 1.4, we have a G-module isomorphism $(P \otimes \mu_\ell) \cong K_2(R)/\ell$. Since $K_2(R)/\ell^G \cong K_2(\mathbb{Z}[1/\ell])/\ell = 0$, $(P \otimes \mu_\ell)$ has no $\mathbb{Z}/\ell$ summands — and hence P contains no summands isomorphic to $\mu_\ell^{\otimes -1}$.

Finally, we have $(P \otimes \mu_\ell^{\otimes 2}) \cong K_4(R)/\ell$ and $(P \otimes \mu_\ell^{\otimes 3}) \cong K_6(R)/\ell$ by 6.5. Again, the transfer argument shows that $K_n(R)/\ell^G \cong K_n(\mathbb{Z}[1/\ell])/\ell$ for $n = 4, 6$. These groups are known to be zero by [R4] and [EGS]; see 1.9. It follows that P contains no summands isomorphic to $\mu_\ell^{\otimes -2}$ or $\mu_\ell^{\otimes -3}$.

Vandiver's conjecture 8.5. If ℓ is an irregular prime then $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}])$ has no ℓ-torsion. Equivalently, the natural representation of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ on $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$ is a sum of G-modules $\mu_\ell^{\otimes i}$ with i odd.

This means that complex conjugation c acts as multiplication by -1 on the ℓ-primary subgroup of $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])$, because c is the unique element of G of order 2.

As partial evidence for this conjecture, we mention that Vandiver's conjecture has been verified for all primes up to 12 million; see [Bu12]. We also known from 8.4 that $\mu_\ell^{\otimes i}$ does not occur as a summand of $\mathrm{Pic}(R)/\ell$ for $i = 0, -2$.

Remark 8.5.1. The Herbrand-Ribet theorem [Wash, 6.17–18] states that $\ell | B_k$ if and only if $\mathrm{Pic}(R)/\ell^{[\ell-2k]} \neq 0$. Among irregular primes < 4000, this happens for at most 3 values of k. For example, $37|c_{16}$ (see 7.12), so $\mathrm{Pic}(R)/\ell^{[5]} = \mathbb{Z}/37$ and $\mathrm{Pic}(R)/\ell^{[k]} = 0$ for $k \neq 5$.

Historical Remark 8.5.2. What we now call "Vandiver's conjecture" was actually discussed by Kummer and Kronecker in 1849–1853; Harry Vandiver was not born until 1882 and only made his conjecture circa 1920. In 1849, Kronecker asked if Kummer conjectured that a certain lemma ([Wash, 5.36]) held for all p, and that therefore p never divided h^+ (*i.e.*, Vandiver's conjecture holds). Kummer's reply [Kum, pp.114–115] pointed out that the Lemma could not hold for irregular p, and then called the assertion [Vandiver's conjecture] "a theorem still to be proven." Kummer also pointed out some of its consequences. In an 1853 letter (see [Kum, p.123]), Kummer wrote to Kronecker that in spite of months of effort, the assertion [Vandiver's conjecture] was still unproven.

For the rest of this paper, we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, where $\zeta^\ell = 1$.

Theorem 8.6. *(Kurihara* [Kur]*) Let ℓ be an irregular prime number. Then the following are equivalent for every k between 1 and $\frac{\ell-1}{2}$:*

(1) $\mathrm{Pic}(\mathbb{Z}[\zeta])/\ell^{[-2k]} = 0$.
(2) *$K_{4k}(\mathbb{Z})$ has no ℓ-torsion;*
(3) *$K_{2a(\ell-1)+4k}(\mathbb{Z})$ has no ℓ-torsion for all $a \geq 0$;*
(4) $H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) = 0$.

In particular, Vandiver's conjecture for ℓ is equivalent to the assertion that $K_{4k}(\mathbb{Z})$ has no ℓ-torsion for all $k < \frac{\ell-1}{2}$, and implies that $K_{4k}(\mathbb{Z})$ has no ℓ-torsion for all k.

Proof. Set $P = \mathrm{Pic}(R)/\ell$. By Kummer theory (see 1.4), $P \cong H^2(R, \mu_\ell)$ and hence $P \otimes \mu_\ell^{\otimes 2k} \cong H^2(R, \mu_\ell^{\otimes 2k+1})$ as G-modules. Taking G-invariant subgroups shows that $H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) \cong (P \otimes \mu_\ell^{\otimes 2k})^G \cong P^{[-2k]}$. Hence (1) and (4) are equivalent.

By 6.3, $K_{4k}(\mathbb{Z})/\ell \cong H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1})$ for all $k > 0$. Since $\mu_\ell^{\otimes b} = \mu_\ell^{\otimes a(\ell-1)+b}$ for all a and b, this shows that (2) and (3) are separately equivalent to (4). □

Theorem 8.7. *If Vandiver's conjecture holds for ℓ then the ℓ-primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all k.*

If Vandiver's conjecture holds for all ℓ, the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all k.

(We know that the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all $k < 500$, by 7.12.)

Proof. Set $P = \mathrm{Pic}(R)/\ell$. Vandiver's conjecture also implies that each of the "odd" summands $P^{[1-2k]} = P^{[\ell-2k]}$ of P is cyclic, and isomorphic to $\mathbb{Z}_\ell/c_k$; see [Wash, 10.15] and 3.8.2 above. Since $\mathrm{Pic}(R) \otimes \mu_\ell^{\otimes 2k-1} \cong H^2(R, \mu_\ell^{\otimes 2k})$, taking G-invariant subgroups shows that $P^{[1-2k]} \cong H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k})$. By theorem 6.2, this group is the ℓ-primary torsion in $K_{4k-2}(\mathbb{Z}[1/\ell])$. □

Using 2.10 and 2.11 we may write the Bernoulli number $B_k/4k$ as c_k/w_{2k} in reduced terms, with c_k odd. The following result, which follows from theorems 0.1, 8.6 and 8.7, was observed independently by Kurihara [Kur] and Mitchell [Mit].

Corollary 8.8. *If Vandiver's conjecture holds, then $K_n(\mathbb{Z})$ is given by Table 8.8.1, for all $n \geq 2$. Here k is the integer part of $1 + \frac{n}{4}$.*

$n \pmod 8$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

Table 8.8.1. The K-theory of $\mathbb{Z}$, assuming Vandiver's Conjecture.

Remark 8.9. The elements of $K_{2i}(\mathbb{Z})$ of odd order become divisible in the larger group $K_{2i}(\mathbb{Q})$. (The assertion that an element a is divisible in A means that for every m there is an element b so that $a = mb$.) This was proven by Banaszak and Kolster for i odd (see [Ban, thm. 2]), and for i even by Banaszak and Gajda [BG, Proof of Prop.8]. It is an open question whether there are any divisible elements of even order.

For example, recall from 7.12 that $K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ and $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$. Banaszak observed [Ban] that these groups are divisible in $K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Q})$, *i.e.*, that the inclusions $K_{22}(\mathbb{Z}) \subset K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Z}) \subset K_{30}(\mathbb{Q})$ do not split.

Let t_j and s_j be respective generators of the summand of $\mathrm{Pic}(R)/\ell$ and $K_1(R; \mathbb{Z}/\ell)$ isomorphic to $\mu_\ell^{\otimes -j}$. The following result follows easily from 6.4 and 8.2, using the proof of 8.3, 8.6 and 8.7. It was originally proven in [Mit]; another proof is given in the article [MKH] in this Handbook. (The generators $s_j\beta^j$ were left out in [Mit2, 6.13].)

Theorem 8.10. *If ℓ is an irregular prime for which Vandiver's conjecture holds, then $K_* = K_*(\mathbb{Z}; \mathbb{Z}/\ell)$ is a free module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$ on the $(\ell-3)/2$ generators y_i described in 8.3, together with the generators $t_j\beta^j \in K_{2j}$ and $s_j\beta^j \in K_{2j+1}$.*

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