THE BERNOULLI OPERATOR

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ABSTRACT. This document explores the Bernoulli operator, giving it a variety of different definitions. In one definition, it is the shift operator acting on infinite strings of binary digits. In another definition, it is the transfer operator (the Frobenius-Perron operator) of the Bernoulli map, also variously known as the doubling map or the sawtooth map.

The map is interesting for multiple reasons. One is that the set of infinite binary strings is the Cantor set; this implies that the Bernoulli operator has a set of fractal eigenfunctions. These are given by the Takagi (or Blancmange) curve. The set of all infinite binary strings can also be understood as the infinite binary tree. This binary tree has a large number of self-similarities, given by the dyadic monoid. The dyadic monoid has an extension to the modular group $PSL(2,\mathbb{Z})$, which plays an important role in analytic number theory; and so there are many connections between the Bernoulli map and various number-theoretic functions, including the Moebius function and the Riemann zeta function.

The Bernoulli map has been studied as a shift operator, in the context of functional analysis [24]. More recently, it has been studied in the physics community as an exactly solvable model for chaotic dynamics and the entropy-increaing thermodynamic arrow of time[26, 14, 6].

Some of what is presented here is review material. New content includes a full development of the continuous spectrum of this operator, including a presentation of the fractal eigenfunctions, and how these span the same space as the the Hurwitz-zeta function eigenfunctions. That is, both provide a basis for the operator, and the fractal basis is a linear combination of the smooth basis, and v.v. The connection to both classic and analytic number theory is also underscored. A simple proof that any totally multiplicative function is associated with a function obeying the multiplication theorem is provided.

This document is meant to be read along with multiple companion papers which further expand the ties into number theory, and into fractal symmetries[30]. Most notable of these is that the shift operator acting on continued fractions gives the Gauss-Kuzmin-Wirising (GKW) operator[28]; the (iso-)morphism between the continued fractions and the real numbers given by the Minkowski Question Mark function[33, 29]. This, in turn, allows a novel reformulation of the Riemann Hypothesis as a bound on a rapdily-converging series representation for the Riemann zeta function[11, 31].

An effort is made to keep the presentation as simple as possible, making this document accesible to the general mathematical audience. This document is perennially in draft form, having multiple unfinished sections. It is occasionally updated with new results or clarifications.

1. THE BERNOULLI OPERATOR

THIS IS A SET OF WORKING NOTES. It's somewhat loosely structured, and sometimes messy. The presentation alternates between easy, and advanced: many parts of this paper are readable by students with modest mathematical knowledge, while other sections presume and require expert familiarity. In general, the harder topics appear later in the text.

The intro hasn't been written yet; but if it was, it would explain why this is an interesting topic. In short, having a clear understanding of transfer operators is vital to the branches of physics concerned with dynamical systems and the arrow of time. The Bernoulli operator

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just happens to be a particularly simple, exactly-solvable case. What is more, Hamltonian systems very frequently have a repeller with a stucture that is the product of a Cantor set and a smooth manifold; the dynamics on the Cantor set part of the repeller is then isomorphic to the Bernoulli shift. For example, Sinai's billiards are isomorphic to a Bernoulli shift [xxx need ref]. Geodesic flows on manifolds of negative curvature (Anosov flows) are also isomorphic to the Bernoulli shift [xxx need ref]. This wide applicability justifies an exhaustive review of the topic.

General background material, laying a modern foundation for the study of chaotic dynamical systems and statistical mechanics, can be found in Mackey[21] and Gaspard[15]. Some recent general results for Frobenius-Perron and Koopman operators are given by Ding[5]; see also references therein [xxx todo]. There are many earlier results on the Bernoulli operator, which is known by many names; besides being called a "Bernoulli shift" or "Bernoulli scheme", it is sometimes more prosaically refered to as a "expanding map of the circle". Kolmogorov and Sinai introduced entropy as a means for determining isomorphic shifts[1958, need ref]. Rochberg studies it as a shift operator[24]. Ornstein[need ref, in V.I. Arnold, A. Avez, "Ergodic problems of classical mechanics" , Benjamin (1968)] provides an isomorphism proof, showing when two different Bernoulli shifts are isomorphic to one-another. Ruelle studies it as a prototype for Axiom A systems[26]. Gaspard provides the Bernoulli polynomial solutions[14]. Driebe shows how it is conjugate to a broad class of maps, and pursues generalizations[6].

The layout is roughly as following (xxx this summary needs work)

- Define the Bernoulli operator, demonstrate some of its eigenfunctions and eigenvalues. The presentation is a simplified variant of the material in [6], generalized to p-adic definitions.
- Present the Hurwitz Zeta and fractal eigenfunctions; show that these are just linear combinations of one-another, and span the same space. Notable is the fact that the fractal eigenfunctions are not differentiable on the rational numbers, whereas the Hurwitz Zeta eigenvalues are classically differentiable almost everywhere.
- Present a more abstract, but more firm/well-founded foundation for the earlier results. The more abstract formulation is in terms of the Cantor set. Here, we take the Cantor set to be the Cartesian product $2^\omega = 2 \times 2 \times \cdots$ of a countably infinite number ω of copies of the set $2 = \{0,1\}$. That is, we take the Cantor set to be the set of all infinitely long strings of binary digits, taken with the product topology. Of course, strings of binary digits can represent the real numbers; but the Cantor set differs from the real numbers in that, unlike the reals, one has $0.111\cdots \neq 1.000\cdots$. This foundation can then lead to deeper results, as well as shed light on some of the more confusing/paradoxical results that arise in the informal treatment.
- Show that the Perron-Frobenius operator is the measure-theoretic pullback. This is a nice, formal, category-theoretic way of defining the FP transfer operator.

Some of the sections, including the sections on orthogonality and completeness, are insufficiently rigorous, and could be tightened up a good bit.

Some new results that are general (and not related to Bernoulli) are presented here:

- The paragraph following 8.6 which generalizes the Wold decomposition 7.2, and shows how to build eigenfunctions, in general, for any transfer operator on any Cartesian product space. (It would be nice to reconcile this with the results of Ding[5] as to the domain of the spectrum (a disk, for L_1 spaces, which are the most frequently studied). Now, one of the recurring themes in this set of papers is self-similarity and scaling behavior; such self-similarity will always hold, in general, on uniform Cartesian products; and so many new, general facts could be easily stated here as well.

- The results of section 10 are entirely new, as far as I know, and provide a much clearer vocabulary (to me, at least) with which to talk about transfer operators.

Some new results that are specific to the Bernoulli operator:

- The presentation of the Hurwitz zeta eigenfunctions, of section 7.3, I believe are new and have never been reported before. Notable is that these are well-defined for eigenvalues lying outside of the unit disk $|\lambda| \leq 1$; in fact, these are well-defined for any complex λ. This in no way contradicts general results that the spectrum of Frobenius-Perron operators lies in the unit disk when one considers eigenfunctions that lie in L_1. These eigenfunctions are analytic; they are just not integrable over the entire unit interval. This simply shows: there is more to life than just the spectrum inside the unit disk.
- The presentation of the fractal eigenfunctions, of section 8.1, are new; the observation that these are self-similar and have a scaling behavior is new.

2. Introduction

The method of the transfer operator was introduced by David Ruelle[26][verify ref] as a powerful mechanism for studying the nature of iterated maps. Before this, iterated functions were typically studied by looking at the trajectories of sets of points. This approach, though obvious, has disadvantages for both the disciplines of physics (*e.g.* loss of differentiability) and mathematics (*e.g.* non-measurable sets, axiom of choice). The transfer operator, sometimes called the Perron-Frobenius operator, or the Ruelle-Frobenius-Perron operator, provides a different, stronger setting, by making incorporating the topology of the space that the map acts on. This section provides a gentle introduction to the concept.

In general, an iterated function $g : X \to X$ maps a space X to itself. When studying an iterated function, one often asks about the orbits of points: Are there fixed points, attractors or repellors, or saddle points? Is the action of g may be ergodic or chaotic? Endowing the set X with a topology (for the unit interval, the natural topology on the reals), one typically asks if g is strong or weak mixing, or merely topologically mixing. The above questions may all be grouped under a heading: 'the study of iterated operators by means of point set topology'. A change in perspective is gained by introducing a measure on the space X. This radically alters how g can be discussed.

Consider how g acts on distributions, rather than points. Suppose the space X is the unit interval $[0,1]$. Intuitively, consider a dusting of points on the unit interval, with a local density given by $\rho(x)$ at point $x \in [0,1]$, and then consider how this dusting or density evolves upon iteration by g. This verbal description may be given form as

$$\rho'(y) = \int_0^1 \delta\left(y - g(x)\right) \rho(x)\, dx \tag{2.1}$$

where $\rho'(y)$ is the new density at point $y = g(x)$ and δ is the Dirac delta function. In this viewpoint, g becomes an operator that maps densities ρ to other densities ρ', or notationally,

$$\mathscr{L}_g \rho = \rho'$$

The operator $\mathscr{L}_g$ is called the transfer operator or the Ruelle-Frobenius-Perron operator. It is a linear operator, for clearly

$$\mathscr{L}_g(a\rho_1 + b\rho_2) = a\mathscr{L}_g\rho_1 + b\mathscr{L}_g\rho_2$$

for constants a, b and densities ρ_1, ρ_2.

Although the word "density" implies that ρ is a smooth map from the unit interval to the non-negative reals, there is no particular need to enforce this. The function ρ may be a map from the unit interval to any ring R, and it need not be smooth, differentiable or even continuous. Usually, we'll take the ring R to the reals $\mathbb{R}$, but in fact, it could be any ring or field. Thus, $\mathscr{L}_g$ is an operator that maps functions on the unit interval to other functions on the unit interval:

$$\mathscr{L}_g : \mathscr{F}[0,1] \to \mathscr{F}[0,1]$$

with $\rho \in \mathscr{F}[0,1]$. The precise properties of $\mathscr{L}_g$ depend strongly on what, exactly, is taken to be "the set of functions $\mathscr{F}[0,1]$ on the unit interval". An exploration of the different spaces makes up much of the content of this paper. There are many spaces to choose from: the space of polynomials, the space of square-integrable functions, or possibly yet other spaces. In general, the properties of $\mathscr{L}_g$ depend on the topology given to $\mathscr{F}$. Most generally, $\mathscr{L}_g$ is an operator acting on a topological space endowed with multiplication and addition, that is, a topological vector space.

The definition of eqn 2.1, for a general space X, requires the definition of an integral on X, and the definition of the Dirac delta function. This is in general possible only when X is endowed with a measure and the associated Borel sigma algebra. The space $\mathscr{F}[X]$ then must be a space of of measurable, sigma-additive functions, if eqn 2.1 is to be meaningful. Given this setting, the transfer operator turns out to be one and the same as the push-forward of g on the set of measurable, sigma-additive functions. A formalization and proof of this is given in a much later section. For most of what follows, though, these formalities can be avoided.

When the function g is differentiable, and doesn't have a vanishing derivative, the transfer operator can be defined as

$$[\mathscr{L}_g\rho](y) = \sum_{x:y=g(x)} \frac{\rho(x)}{|dg(x)/dx|} \tag{2.2}$$

where the sum is presumed to extend over at most a countable number of points. It is not at all hard to show that, for a "reasonable" class of differentiable functions, that 2.1 and 2.2 are equivalent definitions.

Because $\mathscr{L}_g$ is a linear operator, all of the usual language of operator theory may be brought to bear on its study. As a start, one looks for its representations and the eigenvectors and eigenspaces associated with each representation. It is typically pertinent to discuss any symmetries and isomorphisms these spaces might have. Thus, for example, if $\mathscr{L}_g$ is a compact operator, then it's spectrum is constrained by well-known theorems[need ref]: the spectrum is discrete, countable, bounded, and, if infinite, has an accumulation point at zero. It is from here whence the transfer operator gets the name 'Perron-Frobenius': insofar as compact operators are 'a lot like' finite matrices, the Perron-Frobenius theorem applies. So, for example, if one chooses the space $\mathscr{F}[X]$ to be the space of polynomials on X, then, for many 'reasonable' g, the operator $\mathscr{L}_g$ is compact. After we narrow the discussion to the Bernoulli operator, we will find that this is exactly the case. Alternately, one may choose $\mathscr{F}[X]$ to be a Banach space, or possibly a Hilbert space of square-integrable functions on X. In this case, it turns out that, for many 'reasonable' g, that the conjugate of $\mathscr{L}_g$ is and isometry and specifically, a shift operator. In this case, it is well-known from the

Wold-von Neumann decomposition that the spectrum of $\mathscr{L}_g$ fills the entire unit disk on the complex plane. We'll provide several constructions for the Bernoulli operator that exhibit this (and more; a spectrum that fills the entire complex plane!).

The composition operator $C_g : \mathscr{F}[X] \to \mathscr{F}[X]$ acts on functions and is defined as $C_g f = f \circ g$. It is the right-inverse to $\mathscr{L}_g$ but usually not the left inverse: $\mathscr{L}_g C_g$ is the identity, but $C_g \mathscr{L}_g$ is not, in general. In physics, the composition operator is known as the Koopman operator[19]. In mathematics, the AMS category 47B33 is used to classify articles pertaining to the study of composition operators.

Both operators have physical significance, and occur frequently in the study of dynamical systems and statistical mechanics. For example, Ruelle has shown that the Fourrier transform of the correlation function for Axiom A dynamical systems is meromorphic on a strip[27]; the poles of the correlation function are now called Ruelle reosnances, and can be shown to correspond to the logarithm of the eigenvalues of the transfer operator. In particular, this is where the term "transfer operator" comes from: in spin-lattice systems, the transfer matrix simply describes how nearest-neighbor states are connected. When the interaction is long-range, the matrix becomes infinite-dimensional, and thus an operator. That it also typically happens to satisfy the provisions of the Frobenius-Perron theorem is where the name 'Frobenius-Perron operator comes from. Other physically significant constructions follow, including the resolvant, the Fredholm determinant and the dynamical zeta function.

The Cantor set, and more generally, the product topology on infinite Cartesian products, plays a special role in what follows. This is in part due to the nature of shift operators and the Wold decomposition, but can be considered a generalization. To keep things simple, consider the unit interval. For any $x \in [0,1]$, one may write the decimal expansion for x:

$$x = 0.d_1 d_2 d_3 \cdots = \sum_{n=1}^{\infty} d_n 10^{-n}$$

where d_n is the n'th decimal digit of x. The space of all such decimal expansions is the Cartesian product $D \times D \times D \times \cdots$ where D is the finite set $D = \{0,1,2,3,\cdots,9\}$. Although this is 'obvious', it is not commonly appreciated that this infinite Cartesian product is the Cantor set. This has practical implications for the study of transfer operators: one may choose to work with not only the 'natural' topology on the unit interval, but also with the Cantor set topology (aka the product topology or 'cylinder set' topology). This can have a subtle effect on the space of measurable functions $\mathscr{F}[X]$, and can, as well, open up new ways of thinking about the problem, thus offering additional tools, techniques and theorems.

To be more precise, one considers the unit interval $[0,1]$ to consist of the set of strings

$$\Omega = \left\{ \sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots) : \sigma_k \in \{0, 1, \ldots, p-1\}, x = \sum_{k=0}^{\infty} \sigma_k p^{-(k+1)}, x \in [0,1] \right\}$$

Intuitively, this set is simply the set of all the digits of a base-p expansion of the real numbers $x \in [0,1]$. That one may consider a shift operator on this space is now intuitively obvious; that this shift operator has anything at all to do with that of the Wold decomposition is less so. This connection will be explored in greater detail in this text. This space is equivalent to the collection of all field configurations of a one-dimensional, one-sided lattice, where each lattice location can take on one of p values. In physics, such lattices are called 'spin lattices' or 'spin networks'; this cross-over of concepts provides another way of gaining insight.

A few more remarks remain in order. The space Ω may be endowed with one of two different metrics: the natural metric, the closure of which leads to the natural topology on the real numbers, and the p-adic metric, whose closure gives the p-adic numbers. These are important to the study of finite function fields and p-adic analysis in general. Although this can be quite interesting, this text does not currently deal any further with the p-adic metric or the resulting closure.

The real numbers also have yet another product representation. By considering $\Omega = \mathbb{N}\times\mathbb{N}\times\cdots$, one has sequences of positive integers that may be interpreted as a continued fraction:

$$x = [0;\sigma_1,\sigma_2,\sigma_3,\ldots] = \frac{1}{\sigma_1 + \frac{1}{\sigma_2+\frac{1}{\sigma_3+\ldots}}}$$

The measure on this product space is radically different than the usual measure, and is considerably harder to study; it is the Minkowski measure, the derivative of the Minkowski question-mark function. An introduction to lattice models, the product topology, and their applications to the study of transfer operators for continued fractions may be found in [23]. Another exploration with certain specific results can be found in [33]. The continued fraction will not be further addressed in this text; see instead [28] for a detailed treatment.

XXX ToDo: in order for the Perron-Frobenius thm to apply, we must show that $\mathscr{L}_g$ is non-negative; we have not done so yet.

3. Transfer Operator of the Bernoulli Map

The Bernoulli map is important in physics as an exactly solvable example of deterministic chaos, and, in particular, as a model of symbolic dynamics on Axiom A dynamical systems[26, 14, 6]. A purely mathematical treatment of the operator, considered as a shift operator on Hilbert space, is given in [24]. This and the next section recaps those results using a simplified development and simpler tools. The simplified development enables the discussion of more complex scenarios, given in later sections.

The Bernoulli map is given by

$$b(x) = 2x - \lfloor 2x \rfloor \tag{3.1}$$

where $\lfloor x \rfloor$ denotes the integer part of x. The map can be thought of as popping the leading digit off of the binary or 2-adic expansion of x. This map has a positive Lyapunov exponent and is highly chaotic, as, in a certain sense, one can say that the digits of the binary expansion of some 'arbitrary' number are 'unpredictable', and that the orbits of two close-by numbers will eventually become 'uncorrelated' (after suitably defining what we mean by 'arbitrary' and 'unpredictable'). Closely related is the p-adic map, given by

$$a(x) = px - \lfloor px \rfloor$$

for p an integer. As above, this map has the effect of popping off the leading digit of the base-p expansion of x. In a similar vein, one may also consider the Gauss map

$$g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

which has the effect of lopping of the leading digit of the continued fraction expansion of x. What these three maps have in common is that they deal with different (and, in a certain sense, inequivalent) representations of the continuum of real numbers. Each map is chaotic, but in a different way.

The Ruelle-Frobenius-Perron operator or transfer operator of the Bernoulli map is given by

$$[\mathscr{L}_B f](x) = \frac{1}{2}\left[f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right] \tag{3.2}$$

which follows directly from equation 2.2. Similarly, the transfer operator for the general p-adic map is

$$[\mathscr{L}_p f](x) = \frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right) \tag{3.3}$$

while that for the Gauss map is

$$[\mathscr{L}_G f](x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f\left(\frac{1}{1+x}\right)$$

This last is known as the Gauss-Kuzmin-Wirsing operator[28, 23, 12, 10]. It is not studied further here, although it has bearing on some results, due to its relationship to the representation of the real numbers.

One suggestive aspect of eqn Transfer Operator of the Bernoulli Map is that it takes the suggestive form of the 'multiplication theorem' (see wikipedia).

The Bernoulli operators commute: $\mathscr{L}_p\mathscr{L}_q = \mathscr{L}_q\mathscr{L}_p$ and are multiplicative: $\mathscr{L}_p\mathscr{L}_q = \mathscr{L}_{pq}$. The last property suggests that it is "sufficient" to work with prime p.

As noted above, the properties of the operator $\mathscr{L}_B$ depend on the function space on which it acts, and the topology to be used on the unit interval. In particular, the spectrum of eigenvalues and eigenvectors for $\mathscr{L}_B$ can vary from being discrete, to being continuous, depending on the the function space and topology. The simplest case assumes the natural topology of the reals on the unit interval, and takes as the function space the set of orthogonal polynomials on the unit interval. In this case, the eigenfunctions may be shown to be the Bernoulli polynomials $B_n(x)$, associated with the eigenvalues 2^{-n}. That is, one finds that

$$[\mathscr{L}_B B_n](x) = \frac{1}{2^n} B_n(x)$$

where the first few $B_n(x)$ are

$$\begin{aligned}
B_0(x) =& 1 \\
B_1(x) =& x - \frac{1}{2} \\
B_2(x) =& x^2 - x + \frac{1}{6} \\
B_3(x) =& x^3 - \frac{3x^2}{2} + \frac{x}{2}
\end{aligned}$$

and so on. Perhaps the easiest proof that these are the eigenfunctions may be obtained by considering the generating function for the Bernoulli polynomials:

$$G(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}$$

It is then straight-forward to verify that

$$[\mathscr{L}_B G](x,t) = \frac{1}{2}\left[G\left(\frac{x}{2},t\right) + G\left(\frac{x+1}{2},t\right)\right] = G\left(x,\frac{t}{2}\right)$$

or equivalently, by applying the linearity of $\mathscr{L}_B$, that

$$\left[\mathscr{L}_B G\right](x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\mathscr{L}_B B_n\right](x) = \sum_{n=0}^{\infty} \left(\frac{t}{2}\right)^n \frac{B_n(x)}{n!}$$

and then equating the coefficients of the powers of t. This derivation follows for the general p-adic case[14]:

Theorem. *The eigenvalues of the p-adic transfer operator are the Bernoulli polynomials, and are associated with the eigenvalues p^{-n}. That is, one has*

$$\left[\mathscr{L}_p B_n\right](x) = \frac{1}{p^n} B_n(x)$$

Proof. The proof proceeds as in the above 2-adic case, and hinges on the factorization

$$w^p - 1 = (w-1)\left(w^{p-1} + w^{p-2} + \cdots + 1\right)$$

Here, take $w^p - 1 = e^t - 1$, so that

$$\begin{aligned} \left[\mathscr{L}_p G\right](x,t) &= \frac{1}{p} \sum_{k=0}^{p-1} G\left(\frac{x+k}{p}, t\right) \\ &= \frac{1}{p} \frac{t}{e^t - 1} \sum_{k=0}^{p-1} \exp\left(\frac{(x+k)t}{p}\right) \\ &= \frac{t}{p} \frac{\left(1 + e^{t/p} + e^{2t/p} + \cdots + e^{(p-1)t/p}\right)}{e^t - 1} \\ &= G\left(x, \frac{t}{p}\right) \end{aligned}$$

Then, equating coefficients of powers of t of the generating function, one obtains the desired result. ☐

The above theorem is a fancy restatement of an old and well-known result on the Bernoulli polynomials, namely, the so-called “multiplication theorem” given by Joseph Ludwig Raabe in 1851[need ref] (see wikipedia). This is usually given in the more prosaic form of

$$B_n(px) = p^{n-1} \sum_{k=0}^{p-1} B_n\left(x + \frac{k}{p}\right)$$

but amounts to the same thing. Curiously, multiplication theorems exist not only for the Bernoulli polynomials, but more broadly, including the Gamma function and the Hurwitz zeta function. A later section will give a more general set of eigenfunctions in terms of the Hurwitz zeta. Several other connections to number theory are worth pointing out. Sums of the above kind are not atypical in the study of automorphic and modular forms; indeed, the Hecke operator[2] of number theory is essentially the sum of the Bernoulli operator and its transpose. This is perhaps not accidental: later sections will construct fractal eigenfunctions of $\mathscr{L}_p$ that have self-similarity under the dyadic monoid, which is a certain subset of the modular group. The mathematician’s word “automorphic” is a synonym for the physicist’s “self-similar”.

The following sections provide a more abstract and belabored derivation of the above result. The goal of the abstraction is to develop the machinery needed to explore the Bernoulli operator in more general topological settings. Rather than starting with the Hilbert space of orthogonal polynomials on the unit interval, the next section defines a Banach space on monomials, and its dual. The subsequent section then exposes the matrix elements of the Bernoulli operator in this space.

4. The Polynomial Representation

The reason for working with polynomials is that they provide a convenient way of working with infinitely-differentiable functions. A polynomial, with a finite number of terms, is inherently bounded on the unit interval. This is not generally the case when working with a square-integrable Hilbert space; square-integrable functions can be badly discontinuous, nor are they, in general, bounded. The hand-waving physics behind this is that "nature is smooth", and so, insofar as the operator is a model for some natural physical process, one wants to stick to smooth functions. As will be seen, this has a tremendous effect on the spectrum of the operator.

The polynomial eigenvectors of the Bernoulli operator can be derived in several ways. One seemingly reasonable initial approach is to start with a Hilbert space of orthogonal polynomials on the unit interval. This approach is taken by Driebe[6], who starts with the Legendre polynomials, rescaled to the unit interval, and obtains the resulting Bernoulli eigenvectors. This allows for a derivation of the right eigenvectors, but is then discovered to generate difficulties when considering the left eigenvectors. This arises because the the Bernoulli operator is not invertible in this Hilbert space: it is quite singular, in that $[\mathscr{L}_B f] = 0$ whenever f is skew about 1/2, that is, whenever $f(y) = -f(1/2+y)$. A set of left eigenvectors can be found, but these are not polynomials or even ordinary real-valued functions; rather, they are generalized functions, expressed as derivatives of the Dirac delta function. In essence, a "singular value decomposition" is thus performed on the Bernoulli operator, but, as the operator is not 'nice' or 'symmetric' in any common sense, the left and right factors don't have any particular structure or relation to one-another. From this exercise, one concludes that starting with a Hilbert space of orthogonal polynomials offers no particular benefit.

Generalized functions are linear functionals from a space of functions to the reals. That is, generalized functions belong to the dual space of a function space. Given that the left eigenvectors of the Bernoulli operator are "naturally" found to belong to such a dual space (but only after laborious calculation, as in [6]), it then makes sense to work with these, from the start. Thus, for the following, an infinite-dimensional vector space will be constructed, with the basis elements being the monomials. The generalized functions appear naturally as elements of the dual space. The transfer operator then has a clear and simple representation in this space. In principle, this infinite-dimensional vector space may be taken to be a Banach space; however, the development below does not make any particular use of the norm that Banach spaces are equipped with.

Consider an infinite-dimensional vector space V with a countable ordered set of linearly independent basis vectors e_k labelled by the natural numbers k. A general element $v \in V$ may be written as $v = \sum_{k=0}^{\infty} a_k e_k$. The dual space V^* is the set of all linear functionals $L : V \to \mathbb{R}$. General elements of the dual space may be written as linear combinations of the basis elements e_k^*, which are the maps such that $e_j^*(e_k) = \delta_{jk}$.

For the space $\mathscr{P}$ of real-analytic functions on the unit interval, there are many choices one could make for the basis vectors. Most obviously, basis elements may be taken to be the monomials $e_k = x^k$, so that a general real-analytic function $f \in \mathscr{P}$ is written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.1}$$

This may be trivially interpreted as nothing more than the Taylor's series for a real-valued function expanded at $x = 0$. Other possibilities for basis vectors will be considered later.

The correct formalism for discussing the dual space is a bit trickier. Formally, elements of the dual space are called 1-forms or differentials. Following historical convention, the linear functionals of the dual space may be represented as integrals. That is, the linear functional $L : \mathscr{P} \to \mathbb{R}$ may be written as the integral

$$\int_0^1 l(x)\, f(x)\, dx$$

where $f \in \mathscr{P}$ and l is some "generalized function" or "Schwarz function", such as the Dirac delta function. The difficulty with this is, of course, that "generalized functions" are not, in general, differentiable and can have other surprising properties. However, with care, these can be handled in a consistent manner; a quick review follows, as such use is not uncommon. The dual vectors may be written as

$$e_k^* = \frac{(-1)^k}{k!} \delta^{(k)}(x)$$

where $\delta(x)$ is the Dirac delta function. Thus, one has as the duality relation

$$e_j^*(e_k) = \int_0^1 x^k \, \frac{(-1)^j}{j!} \delta^{(j)}(x)\, dx = \int_0^1 \delta(x)\, \frac{d^j x^k}{dx^j}\, dx = \delta_{jk} \tag{4.2}$$

after integration by parts. The relation 4.2 above demonstrates orthogonality; one may also show completeness:

$$\sum_{n=0}^{\infty} e_n \otimes e_n^* = \sum_{n=0}^{\infty} x^n \, \frac{(-1)^n}{n!} \delta^{(n)}(y) = \delta(x-y) \tag{4.3}$$

in that $f(x) = \int \delta(x-y)\, f(y)\, dy$ whenever f is expressible as an analytic function.

Another common notation is that of quantum mechanical bra-ket or Dirac notation. Here, we write $|n\rangle = e_n$ and $\langle n| = e_n^*$. The orthogonality condition becomes $\langle n|\, m\rangle = \delta_{nm}$ while for completeness one writes

$$\mathbb{I} = \sum_{n=0}^{\infty} e_n \otimes e_n^* = \sum_{n=0}^{\infty} |n\rangle \langle n| \tag{4.4}$$

The re-introduction of the coordinate x is done by writing $\langle x|\, m\rangle = x^m$, while for the transpose one writes $\langle n|\, x\rangle = (-1)^n \delta^{(n)}(x)/n!$. The advantage of the bra-ket notation over the use of the e_k is that it can be used to make clear when one is discussing the coordinate representation, involving x or $\delta(x)$, and when one is discussing the vector space elements in the abstract, without reference to the coordinate representation. To fully articulate this notation, one may write the Taylor's series as

$$\begin{aligned} f(x) &= \langle x|f\rangle \\ &= \sum_{n=0}^{\infty} \langle x|n\rangle \langle n|f\rangle \\ &= \sum_{n=0}^{\infty} x^n \langle n|f\rangle \\ &= \sum_{n=0}^{\infty} x^n \int dy \langle n|y\rangle \langle y|f\rangle \\ &= \sum_{n=0}^{\infty} x^n \int dy (-)^n \frac{\delta^{(n)}(y)}{n!} f(y) \\ &= \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} \end{aligned} \tag{4.5}$$

This notation allows the matrix elements $U_{mn} = \langle m|\mathscr{L}|n\rangle$ of the transfer operator to be articulated as well. Starting with the matrix notation of equation 4.1, one has

$$[\mathscr{L}f](x) = \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} U_{mn} a_n$$

or, equivalently

$$[\mathscr{L}f](x) = \langle x|\mathscr{L}f\rangle = \sum_{m=0}^{\infty} \langle x|m\rangle \langle m|\mathscr{L}f\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle x|m\rangle \langle m|\mathscr{L}|n\rangle \langle n|f\rangle$$

where $\langle n|f\rangle = a_n$ and $\langle n|\mathscr{L}f\rangle = b_n$. Equating each power of x^m one finds, in the matrix notation, that

$$\frac{1}{m!} \left.\frac{d^m [\mathscr{L}f](x)}{dx^m}\right|_{x=0} = \sum_{n=0}^{\infty} U_{mn} \frac{1}{n!} \left.\frac{d^n f(x)}{dx^n}\right|_{x=0}$$

where

$$\langle n|f\rangle = a_n = \frac{1}{n!} \left.\frac{d^n f(x)}{dx^n}\right|_{x=0}$$

and so on. The formulation of the transfer operator given in equation 2.1 fits in this framework as well; it is merely the spatial representation of the matrix elements:

$$\begin{aligned} \delta(x-g(y)) &= \mathscr{L}(x,y) \\ &= \langle x|\mathscr{L}|y\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle x|m\rangle \langle m|\mathscr{L}|n\rangle \langle n|y\rangle \end{aligned}$$

So far, we've avoided discussions of convergence and completeness; these will be returned to shortly. Armed with these three different styles of notation, we can explore the matrix elements of the Bernoulli operator in this polynomial basis.

5. The Bernoulli Operator in the Polynomial Basis

This section reviews the structure of the Bernoulli operator $\mathscr{L}_B$ in the polynomial basis developed above. The matrix elements in the monomial basis are given by

$$[\mathscr{L}_B]_{mk} \equiv U_{mk} \equiv \langle m|U|k\rangle = \frac{\delta_{mk}}{2^m} + \binom{k}{m} \frac{\Theta_{mk}}{2^{k+1}} \tag{5.1}$$

where $\left(\begin{array}{c} m \\ k \end{array} \right) = \frac{m!}{k!(m-k)!}$ is the binomial coefficient, and

$$\Theta_{mk} = \begin{cases} 0 \text{ if } k \leq m \\ 1 \text{ if } k > m \end{cases}$$

is a traceless, pure upper-triangular matrix. These matrix elements are easily obtained by direct substitution, that is, by contemplating the coefficient to the x^m term in the expansion of

$$\left[\mathscr{L}_B x^k\right] = \frac{1}{2}\left[\left(\frac{x}{2}\right)^k + \left(\frac{1+x}{2}\right)^k\right]$$

Both the diagonal elements, and the reason for the appearance of the binomial coefficients should be immediately clear. Note that the evaluation of these matrix elements does not require the evaluation of any sums with an infinite number of non-zero terms; the sums are all finite.

The eigenvalues may be promptly read off the diagonal; these eigenvalues are $\lambda_n = 2^{-n}$. Because the matrix is upper-triangular, it is easily solvable for both the left and right eigenvectors, which agree perfectly with those given by Driebe[6]. Visually, the upper-left of this matrix looks like

$$U_{mk} = \left[\begin{array}{cccccc} 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{16} & \frac{1}{8} & \\ 0 & 0 & \frac{1}{4} & \frac{3}{16} & \frac{3}{16} & \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \\ 0 & 0 & 0 & 0 & \frac{1}{16} & \\ \cdots & & & & & \cdots \end{array} \right]$$

The right and left eigenvectors are developed in the following sections. Note that U_{mk} is very well-behaved: all entries remain small. For large k, each column resembles a Gaussian (this follows from Stirling's approximation for the factorial), and specifically, this Gaussian has a peak at $k = 2m$. That is, one has $U_{mk} \leq U_{m,2m} \approx 1/2\sqrt{\pi m}$ for large m.

5.1. **Right eigenvectors.** The right eigenvectors are denoted by $|B_n\rangle$ and have the vector components

$$\langle k | B_n \rangle = \left(\begin{array}{c} n \\ k \end{array} \right) \left(1 - \Theta_{n,k}\right) B_{n-k} = \begin{cases} \left(\begin{array}{c} n \\ k \end{array} \right) B_{n-k} & \text{for} \quad k \leq n \\ 0 & \text{for} \quad k > n \end{cases} \tag{5.2}$$

where B_k are the Bernoulli numbers. Note that only a finite number of these vector components are non-vanishing.

Theorem. *The $|B_n\rangle$ are eigenvectors of U_{mk}, associated with eigenvalues 2^{-n}. That is,*

$$\mathscr{L}_B |B_n\rangle = \frac{1}{2^n} |B_n\rangle$$

Proof. This may be verified in the monomial basis through brute-force multiplication of the vector into the matrix:

$$\begin{aligned}\sum_{k=0}^{\infty}\langle m|U|k\rangle\langle k|B_n\rangle &= \sum_{k=m}^{n}\left[\frac{\delta_{mk}}{2^m}+\left(\begin{array}{c}k\\m\end{array}\right)\frac{\Theta_{mk}}{2^{k+1}}\right]\left(\begin{array}{c}n\\k\end{array}\right)B_{n-k}\\ &= \frac{1}{2^m}\left(\begin{array}{c}n\\m\end{array}\right)B_{n-m}+...non-trivial-taylor-expn\\ &= \frac{1}{2^n}\langle m|B_n\rangle\end{aligned}$$

XXX To-do: fill in details. Note that this proof does not require the evaluation of any sums with an infinite number of non-zero elements; all sums are finite. □

The right eigenvectors can be given a representation in coordinate space, and these are found to be the Bernoulli polynomials discussed previously:

$$\sum_{k=0}^{\infty}\langle x|k\rangle\langle k|B_n\rangle=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_{n-k}x^k=B_n(x)$$

These were previously shown to be eigenvectors, and so this is consistent with the above. One may conclude that the coordinate representation and the matrix representation for this transfer operator are consistent.

5.2. **Left Eigenvectors.** This matrix expression for U_{mn} also admits left eigenvectors which can be given an explicit representation. Letting the left eigenvectors be denoted by $\left\langle\tilde{B}_n\right|$, they have, for $n>0$, the components

$$\left\langle\tilde{B}_n\middle|k\right\rangle=\left(\begin{array}{c}k\\n-1\end{array}\right)\frac{1}{n}\left(\delta_{nk}+\Theta_{nk}\right) \tag{5.3}$$

The zeroth left eigenvector is a special-case; it has components $\left\langle\tilde{B}_0\middle|k\right\rangle=1/(k+1)$. Unlike the right eigenvectors, the left eigenvectors all have an infinite number of non-zero components. They share the same eigenvalue spectrum with the right eigenvectors, so that

$$\sum_{k=0}^{\infty}\left\langle\tilde{B}_n\middle|m\right\rangle U_{mk}=\frac{1}{2^n}\left\langle\tilde{B}_n\middle|k\right\rangle$$

This may again be demonstrated by a brute-force evaluation of the sum. Despite the fact that an infinite number of the left eigenvector components are non-vanishing, this sum will only contain a finite number of non-zero terms, and thus its finiteness is guaranteed on algebraic grounds.

One may write down a coordinate-space representation for the left eigenvectors, by contracting them against the dual-space basis elements $\langle k|x\rangle$. This leads directly to the generalized functions:

$$\begin{aligned}\left\langle\tilde{B}_n|x\right\rangle &= \sum_{k=0}^{\infty}\left\langle\tilde{B}_n\middle|k\right\rangle\langle k|x\rangle\\ &= \sum_{k=0}^{\infty}\left\langle\tilde{B}_n\middle|k\right\rangle(-)^k\frac{\delta^{(k)}(x)}{k!}\\ &= \frac{1}{n}\sum_{k=n}^{\infty}\left(\begin{array}{c}k\\n-1\end{array}\right)(-)^k\frac{\delta^{(k)}(x)}{k!}\\ &= \frac{(-1)^{n+1}}{n!}\left[\delta^{(n-1)}(1-x)-\delta^{(n-1)}(x)\right] \qquad (5.4)\end{aligned}$$

for the $n>0$ case. The $n=0$ left eigenvector is best understood by integrating it over some arbitrary function $f(x)$:

$$\begin{aligned}\int_0^1 dx\,\left\langle \tilde{B}_0\middle|x\right\rangle f(x) &= \int_0^1 dx\, f(x)\sum_{k=0}^{\infty}(-)^k\frac{\delta^{(k)}(x)}{(k+1)!}\\ &=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{(k+1)!}\\ &=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\int_0^1 x^k dx\\ &=\int_0^1 f(x)dx\\ &=\left\langle \tilde{B}_0\middle|f\right\rangle\end{aligned}$$

The other left eigenvectors can also be made more concrete by looking at how they act on some function $f(x)$; this may be written as

$$\left\langle \tilde{B}_n\middle|f\right\rangle = \frac{1}{n!}\left[f^{(n-1)}(1)-f^{(n-1)}(0)\right] \tag{5.5}$$

5.3. **Duality.** That the left eigenvectors are dual to the Bernoulli polynomials, which can be verified in either the matrix-element basis, or the coordinate-space representation.

Theorem. *The left and right eigenvectors are dual, in that*

$$\left\langle \tilde{B}_n\middle|B_m\right\rangle = \sum_{k=0}^{\infty}\left\langle \tilde{B}_n\middle|k\right\rangle\langle k|B_m\rangle = \delta_{nm}$$

and furthermore, the duality is algebraic, in that no infinite sums need be performed to demonstrate duality.

Proof. Consider first the $n=0$ case. One has

$$\begin{aligned}\left\langle \tilde{B}_0\middle|B_m\right\rangle &= \sum_{k=0}^{\infty}\left\langle \tilde{B}_0\middle|k\right\rangle\langle k|B_m\rangle\\ &= \sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}B_{m-k}\\ &= \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\\ &= \delta_{m0}\end{aligned}$$

where the substitution $j=m-k$ was made to obtain the last sum. The vanishing of the last sum is a well-known identity on the Bernoulli numbers. The $n\neq 0$ case requires more work, but ends similarly:

$$\begin{aligned}
\left\langle \tilde{B}_n \middle| B_m \right\rangle &= \sum_{k=0}^{\infty} \left\langle \tilde{B}_n \middle| k \right\rangle \langle k | B_m \rangle \\
&= \sum_{k=0}^{m} \binom{k}{n-1} \frac{1}{n} (\delta_{nk} + \Theta_{nk}) \binom{m}{k} B_{m-k} \\
&= \frac{1}{n} \binom{n}{n-1} \binom{m}{n} B_{m-n} + \sum_{k=n+1}^{m} \binom{k}{n-1} \frac{1}{n} \binom{m}{k} B_{m-k} \\
&= \begin{cases} 0 & \text{for } n > m \\ B_0 & \text{for } n = m \\ \binom{m}{n} B_{m-n} + \frac{m!}{n!(m-n+1)!} \sum_{k=n+1}^{m} \binom{m-n+1}{m-k} B_{m-k} & \text{for } n < m \end{cases}
\end{aligned}$$

The last case can be shown to vanish, by again making the substitution $j = m - k$, to get

$$\begin{aligned}
\binom{m}{n} B_{m-n} &+ \frac{m!}{n!(m-n+1)!} \sum_{j=0}^{m-n-1} \binom{m-n+1}{j} B_j \\
&= \frac{m!}{n!(m-n+1)!} \sum_{j=0}^{m-n} \binom{m-n+1}{j} B_j = 0
\end{aligned}$$

thus concluding the proof. Notice that this proof does not require the evaluation of any infinite sums: all sums are performed over a finite number of terms. □

This proof of duality may also be conducted in coordinate space, where it takes the form

$$\begin{aligned}
\left\langle \tilde{B}_n \middle| B_m \right\rangle &= \int_0^1 \left\langle \tilde{B}_n \middle| x \right\rangle \langle x | B_m \rangle \, dx \\
&= \int_0^1 \frac{(-1)^{n+1}}{n!} \left[\delta^{(n-1)}(1-x) - \delta^{(n-1)}(x) \right] B_m(x) \, dx \\
&= \begin{cases} 0 & \text{for } m < n-1 \\ \frac{m!}{n!(m-n+1)!} \left[B_{m-n+1}(1) - B_{m-n+1}(0) \right] & \text{for } m \geq n-1 \end{cases} \\
&= \delta_{mn} \qquad (5.6)
\end{aligned}$$

and so, as with most of the previous results, one may be lulled into a sense of complacency about the equivalence of the coordinate-space and monomial-vector-space representations. This complacency is ill-founded, as demonstrated below.

5.4. **Completeness.** Given this duality, the operator

$$\mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \left\langle \tilde{B}_n \right| \qquad (5.7)$$

can then be recognized as a projection operator. In fact, it is algebraically complete in the monomial basis, in that

$$\langle j | \mathbb{I}_B | k \rangle = \langle j | \sum_{n=0}^{\infty} |B_n\rangle \left\langle \tilde{B}_n \right| |k\rangle = \delta_{jk} \qquad (5.8)$$

holds. This can be shown using essentially the same operations as in the proof above. Also, as before, this demonstration involves sums with only a finite number of terms, and so completeness may be taken as an algebraic property. That is, $\mathbb{I}_B$ may be taken to be the identity operator on the vector space of polynomials.

Curiously, this identity operator $\mathbb{I}_B$ expanded in the Bernoulli basis as in formula 5.7 is the Euler-Maclaurin summation formula in disguise[14, 6]. This may be seen by expanding

$$\begin{aligned} f(x) &= \langle x|f\rangle \\ &= \sum_{m=0}^{\infty} B_m(x)\left\langle \tilde{B}_m\right| f\rangle \\ &= \int_0^1 f(y)\,dy + \sum_{m=0}^{\infty} \frac{B_m(x)}{m!}\left[f^{(m-1)}(1) - f^{(m-1)}(0)\right] \end{aligned} \tag{5.9}$$

This may be compared to the $n=1$ case of the traditional Euler-Maclaurin summation formula,

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k+x}{n}\right) = \int_0^1 f(y)\,dy + \sum_{m=0}^{\infty} \frac{B_m(x)}{n^m m!}\left[f^{(m-1)}(1) - f^{(m-1)}(0)\right]$$

By combining equations 5.75.8 and 5.9 one is sorely tempted to deduce orthogonality over coordinate space. That is, one wants to deduce that

$$\sum_{n=0}^{\infty} \langle x|B_n\rangle\left\langle \tilde{B}_n\right| y\rangle = \delta(x-y) \tag{5.10}$$

which superficially seems to be entirely reasonable. Written this way, however, the equation 5.10 is misleading; it is only valid for functions that are of exponetial type less than 2π; the polynomials belong to this class, and many meromorphic functions, but not, for example, $\sin 2\pi x$. The same holds for the Euler-Maclaurin formula 5.9.

An entire holomorphic function f on the complex plane is said to be of exponential type τ when there exist constants M and τ such that

$$|f(z)| \leq Me^{\tau|z|}$$

as $|z| \to \infty$[3]. Thus, for example, $\sin 2\pi x$ is of exponential type 2π, since it blows up exponentialy along the complex imaginary axis. Any polynomial is of exponential type ε for any $\varepsilon > 0$; thus, again the expansion 5.10 can be safely applied to polynomials. Likewise, $\mathbb{I}_B$ is the identity operator on the space of polynomials, and, more generally on functions of exponential type less than 2π: there are no polynomials (other than $f(x)=0$) that are in the kernel of $\mathbb{I}_B$.

However, as a space, the space of polynomials is not itself complete: the space does not contain all of its limit points. That is, there are sequences of polynomials that converge to infinite series; these infinite series are, by definition, not polynomials, and so are not a part of the space. So, for example, consider the series expansion for the sine wave. Truncated at any finite number of terms, the result is a polynomial. This sequence of ever-higher degree polynomials can be understood to converge to the sine wave (once an appropriate definition of "convergence" is given). However, the sine wave belongs to the kernel of $\mathscr{L}_B$: taking $f(x) = \sin 2\pi x$ in 3.2, one finds that $\mathscr{L}_B \sin 2\pi x = 0$. In fact, one has $\mathscr{L}_B \sin 2\pi nx = 0$ for any odd integer n; and so, in this completed space, $\mathscr{L}_B$ has a rather large kernel. Its just that no polynomial belongs to that kernel; one requires the completion of the space of polynomials by infinite series to find the non-trivial kernel of $\mathscr{L}_B$. Since $\mathbb{I}_B$ is just a projection, the kernel of $\mathbb{I}_B$ and $\mathscr{L}_B$ are identical. Put another way: $\mathbb{I}_B$ has a large, non-trivial kernel on the completed space, while, by contrast, the Dirac delta function

$\delta(x-y)$ only has a trivial kernel. Thus, on the completion of the space, $\mathbb{I}_B \neq \delta(x-y)$, in contradiction to 5.10.

XXX ToDo: tighten up; per Gaspard[14], one completion is the Frechet space of analytic functions of exponential type τ. A Frechet space is a generalization of a Banach space, having a translation-invariant metric.

5.5. **The Bernoulli operator in diagonal form; the Koopman operator.** From the above manipulations, one may deduce that, in the polynomial representation, the Frobenius-Perron operator of the Bernoulli map is

$$\mathscr{L}_B = \sum_{n=0}^{\infty} |B_n\rangle \lambda_n \langle \tilde{B}_n|$$

We can make use of this diagonal form to easily compute formal expressions involving $\mathscr{L}_B$. Thus, for a function $f(x)$ that is expressible as a polynomial series in x, one may write the operator

$$f(\mathscr{L}_B) = \sum_{n=0}^{\infty} |B_n\rangle f(\lambda_n) \langle \tilde{B}_n|$$

whose matrix elements can be explicitly demonstrated in the monomial basis:

$$\langle j|f(\mathscr{L}_B)|k\rangle = \sum_{j\leq n\leq k} \left(\begin{array}{c} n \\ j \end{array}\right) \left(\begin{array}{c} k \\ n-1 \end{array}\right) \frac{B_{n-j}}{n} f\left(2^{-n}\right)$$

As in previous cases, note that the summation involves only a finite number of terms, and is thus manifestly finite (provided that f is finite). As curious example, one may write, $\mathscr{L}_B = \exp(-H_B)$ so that $H_B = -\log \mathscr{L}_B$ has matrix elements

$$\langle j|H_B|k\rangle = \frac{\log(2)}{k+1} \left(\begin{array}{c} k+1 \\ j \end{array}\right) \sum_{m=0}^{k-j} \left(\begin{array}{c} k-j+1 \\ m \end{array}\right) (j+m) B_m$$

None of the eigenvalues λ_n are zero. In the previous section, it was shown that the kernel of $\mathbb{I}_B$ is trivial. Thus, $\mathscr{L}_B$ is invertible. This inverse is known as the *Koopman operator*[19], and is denoted by $\mathscr{K}_B$:

$$\mathscr{K}_B = \sum_{n=0}^{\infty} |B_n\rangle \frac{1}{\lambda_n} \langle \tilde{B}_n|$$

By duality and algebraic completeness, one has that the Koopman operator is both a left and a right inverse,

$$\mathscr{L}_B \mathscr{K}_B = \mathscr{K}_B \mathscr{L}_B = \mathbb{I}_B$$

in the polynomial representation. In this representation, one may honestly write $\mathscr{K}_B = \mathscr{L}_B^{-1}$. This will not at all be the case when one considers the Bernoulli operator $\mathscr{L}_B$ acting on the appropriately-defined completions of the space of polynomials, or on other function spaces: it will be seen to have a large and non-trivial kernel, and so it will not be invertible.

5.6. **Change of Basis Recap.** This section simply recaps the previous results. It was seen above that the monomials form a complete set of basis states that can be used to represent polynomials. The operator $\mathbb{I}_M = \sum_{n=0}^{\infty} |n\rangle \langle n|$ can be called the identity operator over the space of polynomials; it has no non-trivial kernel in that space. Here, the subscript M is used to indicate that the identity operator is built from the monomial states. Thus, for the Bernoulli operator $\mathscr{L}_B$, one may confidently write $\mathscr{L}_B = \mathbb{I}_M \mathscr{L}_B \mathbb{I}_M$ which expands to

$$\mathscr{L}_B = \mathscr{L}_B^{\text{Monomial}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m\rangle \langle m|\mathscr{L}_B|n\rangle \langle n|$$

The superscript "Monomial" is a formal label, used only to emphasize the basis in which the operator was expanded in. As seen above, the matrix elements $U_{mn} \equiv \langle m|\mathscr{L}_B|n\rangle$ are upper-triangular.

The operator $\mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \left\langle \tilde{B}_n \right|$ is also the identity operator on the space of polynomials. Using the same trick to write

$$\mathscr{L}_B = \mathscr{L}_B^{\text{Bernoulli}} = \mathbb{I}_B \mathscr{L}_B \mathbb{I}_B = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |B_m\rangle \left\langle \tilde{B}_m \right| \mathscr{L}_B |B_n\rangle \left\langle \tilde{B}_n \right|$$

one finds that $\mathscr{L}_B^{\text{Bernoulli}}$ is an operator which has matrix elements that are non-zero only on the diagonal: $\left\langle B_m \right| U_B |\tilde{B}_n\rangle = \delta_{mn}\lambda_n$.

Consider now the change of basis from the monomial basis to the Bernoulli basis. Explicitly, this change of basis is

$$\begin{aligned} \mathscr{L}_B^{\text{Bernoulli}} &= \sum_{j,k=0}^{\infty} \left|B_j\right\rangle \delta_{jk}\lambda_k \left\langle \tilde{B}_k \right| \\ &= \sum_{j,k,m,n=0}^{\infty} \left|B_j\right\rangle \left\langle \tilde{B}_j \middle| m \right\rangle \left\langle m \middle| \mathscr{L}_B^{\text{Monomial}} \middle| n \right\rangle \langle n|B_k\rangle \left\langle \tilde{B}_k \right| \\ &= \tilde{B} \mathscr{L}_B^{\text{Monomial}} B \end{aligned} \tag{5.11}$$

where the operators $\tilde{B}$ and B show the change of basis:

$$\tilde{B} = \sum_{j,m=0}^{\infty} \left|B_j\right\rangle \left\langle \tilde{B}_j \middle| m \right\rangle \langle m|$$

and

$$B = \sum_{k,n=0}^{\infty} |n\rangle \langle n|B_k\rangle \left\langle \tilde{B}_k \right|$$

It is not hard to work out that $\tilde{B}B = \mathbb{I}_B$ and that $B\tilde{B} = \mathbb{I}_M$ so $\tilde{B}$ is both a left- and right-inverse of B; one may confidently write $B^{-1} = \tilde{B}$ as a two-sided inverse. Although B and $\tilde{B}$ are inverses of one-another, they are in no way orthogonal. The matrix elements of the transpose of an orthogonal operator are equal to those of the inverse; a quick review of equations 5.2 and 5.3 makes it clear that $\left\langle \tilde{B}_k \middle| n \right\rangle \neq \langle n|B_k\rangle$, and so B is not an orthogonal operator. This is not a surprise: orthogonal operators cannot take a diagonal operator and make it triangular.

5.7. **Operator recap.** XXX To-do: Talk about the operator

$$\sum_n \lambda_n e_n \otimes e_n^*$$

in terms of being a compact operator, a nuclear operator, etc. Talk about the order of the operator, Talk about the projective topological tensor product, in the language of Grothendieck. The primary goal here is to justify the matrix-style notations of the previous section. i.e. the matrix operations given above are valid because $\mathscr{L}_B$ is compact. Its even trace-class: one has that $\sum_n |\lambda_n| < \infty$. See also, notes about Frechet space, maybe that should be here ...

6. Topology, Completeness and Orthogonality

The notions of completeness and orthogonality are treated above without any appeal to topology. That is, they are handled with what is essentially an algebraic approach, where all sums are finite and well defined because all sums involve only a finite number of non-zero terms. This was possible in part by construction, and in part by luck: the Bernoulli operator was a solvable, upper-triangular matrix in the infinite vector space whose basis elements are the monomials. A sum over monomials, where only a finite number of terms are non-zero, is a polynomial. To go beyond this, to get to more general functions, such as, for example, a sum over monomials with an infinite number of non-zero elements, requires the introduction of a topology on the infinite vector space, so that limits of Cauchy sequences can be defined and discussed. There are several ways to provide a topology; the straightforward way is to provide the space with a metric topology. A metric topology endows the infinite vector space with a norm, so that the length of a vector can be given, and the distance between vectors defined as the length of the vector difference. Metrics or norms provide a particularly easy way of discussing the convergence of limits.

With this in mind, the question then turns to "what are the interesting topologies?". Before this question is asked in earnest, it is worth illustrating why, exactly, it is an important question, and why the role of topologies needs to be addressed. Some of the difficulties of sticking to a purely algebraic approach are illustrated in this section.

The operator $\mathbb{I}_B = \sum_{n=0}^{\infty} |B_n\rangle \langle \tilde{B}_n|$ was shown to be algebraically complete on the space of polynomials. By this, it is meant that there is no polynomial that lies in the kernel of $\mathbb{I}_B$, other than the trivial polynomial $p(x) = 0$. Equivalently, there does not exist any polynomial for which the identity $p(x) = -p(x+1/2)$ holds: more broadly, there is no such thing as a periodic polynomial. A function which obeys $f(x) = -f(x+1/2)$ is of necessity periodic, of which $f(x) = \sin(2\pi x)$ is a canonical example. More generally, sine and cosine waves which have an odd number of periods in the unit interval are all in the kernel of the Bernoulli operator, when that operator is taken in the coordinate-space. This may be seen very easily simply by direct substitution into equation 3.2, which promptly yields

$$\mathscr{L}_B^{\text{Coordinate}} \sin 2\pi(2k+1)x = \frac{1}{2}\left[\sin \pi(2k+1)x + \sin \pi(2k+1)(x+1)\right] = 0 \tag{6.1}$$

for integers k; likewise for the cosine. Not so for waves with an even number of periods:

$$\mathscr{L}_B^{\text{Coordinate}} \sin 4\pi kx = \frac{1}{2}\left[\sin 2\pi kx + \sin 2\pi k(x+1)\right] = \sin 2\pi kx$$

Here, the superscript "Coordinate" is introduced to distinguish the operator in the coordinate basis, as given in equation 3.2, from the same operator in the monomial basis.

In a previous section, it was established that $\mathscr{L}_B^{\text{Bernoulli}} = \mathscr{L}_B^{\text{Monomial}}$, when these are considered as operators acting on the space of polynomials. However, these are not at all equivalent if one considers them as operators acting on sines and cosines. In particular, consider $\mathbb{I}_B$ acting on $f(x)=\exp 2\pi ikx$:

$$\begin{aligned}
\left[\mathbb{I}_B f\right](x) =& \langle x|\mathbb{I}_B|f\rangle \\
=& \sum_{n=0}^{\infty} \langle x|B_n\rangle \left\langle \tilde{B}_n\right| f\rangle \\
=& \sum_{n=1}^{\infty} \langle x|B_n\rangle \frac{1}{n!}\left[f^{(n-1)}(1)-f^{(n-1)}(0)\right]+\int_0^1 f(x)\,dx \\
=& \sum_{n=1}^{\infty} \langle x|B_n\rangle \frac{1}{n!}\left[(2\pi ik)^{n-1}e^{2\pi ik}-(2\pi ik)^{n-1}\right] \\
=& 0
\end{aligned}$$

where the second step makes use of equation 5.5. This is remarkable, as any periodic wave constructed from sines and cosines seems to be in the kernel. By contrast, $\mathbb{I}_M$ does not behave this way:

$$\begin{aligned}
\left[\mathbb{I}_M f\right](x) =& \langle x|\mathbb{I}_M|f\rangle \\
=& \sum_{n=0}^{\infty} \langle x|n\rangle \langle n|f\rangle \\
=& \sum_{n=0}^{\infty} \langle x|n\rangle \frac{1}{n!} f^{(n)}(0) \\
=& \sum_{n=0}^{\infty} \langle x|n\rangle \frac{1}{n!}(2\pi ik)^n \\
=& \sum_{n=0}^{\infty} x^n \frac{1}{n!}(2\pi ik)^n \\
=& \exp 2\pi ikx
\end{aligned}$$

which is just a re-derivation of equation 4.5. Thus, $\mathbb{I}_B$ and $\mathbb{I}_M$ are inequivalent when acting on sine functions; so $\mathscr{L}_B^{\text{Monomial}}$ and $\mathscr{L}_B^{\text{Bernoulli}}$ are inequivalent as well.

Since the sine function is the limit of a polynomial sequence, it seems strange or somehow contradictory that $\mathscr{L}_B^{\text{Bernoulli}}$ has only a trivial kernel on the space of polynomials, while killing odd-period sine and cosine functions. In order to define limits, or more precisely, in order to define $\sin 2\pi x$ as the limit of a sequence of polynomials, one must define the manner in which a polynomial sequence can converge to a function, and, for that, one must have a topology. This conundrum is revisited, and solved, in section 7.1.

Can this conundrum be escaped without appealing to topology? Since $\mathbb{I}_B$ seems to be somehow incomplete when considering sine functions, perhaps, one might think, this lack of completeness is due to the form of the left eigenstates given in equation 5.4. One might make a guess that perhaps a more truly complete set of states can be found by considering

$$\tilde{S}_n(x)=\frac{(-)^{n+1}}{n!}\left[\delta^{(n-1)}(1-x)+\delta^{(n-1)}(x)\right]$$

so that sums and differences of the duals $\tilde{B}_m(x)$ and $\tilde{S}_n(x)$ can be used to regain the duals to the monomials $\langle n|y\rangle=(-)^n\delta^{(n)}(y)/n!$.

Theorem. *The duals to $\tilde{S}_n(x)$ are given by $S_n(x)=nE_n(x)/2$ where the $E_n(x)$ are the Euler polynomials.*

Proof. Consider the generating function for the Euler polynomials

$$G_E(x,t) = \frac{2e^{xt}}{1+e^t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$$

Then one has, by taking the left hand side,

$$\int_0^1 \tilde{S}_n(x)\,\frac{2e^{xt}}{1+e^t}\,dx = \frac{2}{n}\frac{t^{n-1}}{(n-1)!}$$

and, performing the same operation on the right hand side,

$$\int_0^1 \tilde{S}_n(x)\sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 \tilde{S}_n(x)\,E_k(x)\,dx$$

Then, equating the two sides, one has demonstrated duality of these states:

$$\int_0^1 \tilde{S}_n(x)\,E_k(x)\,dx = \frac{2\delta_{k,n-1}}{n}$$

which completes the proof. □

As with the Bernoulli polynomials, one can, to a limit extent, make a restricted completeness statement in coordinate space. That is, if one decomposes a function $f(y) = \sum_{k=1}^{\infty} a_k S_k(y)$ then one easily finds

$$\int_0^1 \sum_{n=1}^{\infty} S_n(x)\tilde{S}_n(y)\,f(y)\,dy = f(x)$$

from which one wants to conclude, once again incorrectly, or at least misleadingly, that

$$\sum_{n=1}^{\infty} S_n(x)\tilde{S}_n(y) = \delta(x-y)$$

by repeating the same concerns and issues that lead to equation 5.10. The fault is the assumption that arbitrary, non-polynomial $f(y)$ can be decomposed in the fashion given. In fact, the operator $\mathbb{I}_S = \sum_{n=0}^{\infty} |S_n\rangle\langle\tilde{S}_n|$ also has a large kernel: this time, all functions that are evenly periodic are in the kernel. That is, any function for which one has $f(y) = f(y+1/2)$ lies in the kernel of $\mathbb{I}_S$.

One might hope that one can remedy the above situation by taking the sum $\mathbb{I}_C = \mathbb{I}_B + \mathbb{I}_S$ with one operator projecting out the even periodic functions, and the other the odd periodic functions, and that somehow, between the two of them, making a whole. However, one immediately runs into a problem with the basis functions. The $\tilde{S}_n(x)$ are not orthogonal to the $B_n(x)$, and vice-versa. This is easily seen by considering the the generating function for the Bernoulli polynomials:

$$G_B(x,t) = \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

Integrating, one gets

$$\int_0^1 \tilde{S}_n(x)G_B(x,t)\,dx = \frac{t^n}{n!}\frac{e^t+1}{e^t-1} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 \tilde{S}_n(x)\,B_k(x)\,dx$$

However, one finds that $\int_0^1 \tilde{S}_n(x)\,B_k(x)\,dx = 0$ only for $k < n-1$, and similarly for $\int_0^1 \tilde{B}_n(x)\,S_k(x)\,dx$.

More generally, any such attempt to patch up the situation is doomed to fail. In essence, we are looking for a basis such that every point in the space is a linear combination of a finite number of basis vectors. Such a basis is called an algebraic or Hamel basis. But the number of limit points in the space are uncountable, and so the Hamel basis cannot be countable[4]. In such a situation, the basis vectors can't be bi-orthogonal.

7. The Fourier Representation

As explored above, the Bernoulli operator is clearly quite different on the completion of the space of polynomials than it is on the space of polynomials itself. At this point, it would be nice to have a discussion how one might create completions of the space of polynomials, how many of these are distinct, and what their different properties might be. [XXX ToDo, do the above, at least in some appendix or something]. However, at this point, we make an ansatz, and jump to one reasonable space that can fill the bill: the Hilbert space of square-integrable functions on the unit interval. This is a well-known, familiar space widely studied in textbooks, and so is a suitable setting for further analysis of the Bernoulli operator.

In what follows, many of the properties of this space will not be required or appealed to. The only important ingredient we'll make use of is that this space has a basis spanned by the functions $\sin 2\pi nx$ and $\cos 2\pi nx$ for positive integers n. Also, keep in mind that, in general, an element of this space is not differentiable or even continuous (the square wave is a textbook example, although, below, we'll construct the Cantor curve, which is an element of this space, but is discontinuous at all dyadic rationals). Elements of this space are also not bounded, but may diverge to infinity. From the physics viewpoint, this means that this space contains many "non-physical" functions. So, for example, in physical systems, the kinetic energy commonly appears as the square of the derivative; when a function is discontinuous, the kinetic energy is effectively infinite. So, in order to stick to "physical" systems. one should properly have defined some sort of metric on the polynomials that somehow took into account their "kinetic energy", and then used this to define the completion of the space. We save this task for some rainy day.

The general element of the space of square-integrable functions on the unit interval is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + b_n \sin 2\pi nx \tag{7.1}$$

Applying $\mathscr{L}_B$ as defined in eqn 3.2 to this, one immediately finds that

$$\begin{aligned}\left[\mathscr{L}_B f\right](x) = & a_0 + \sum_{n \text{ even}} a_n \cos \pi nx + b_n \sin \pi nx \\ = & a_0 + \sum_{n=1}^{\infty} a_{2n} \cos 2\pi nx + b_{2n} \sin 2\pi nx\end{aligned}$$

The doubling of the Bernoulli map 3.1 manifests itself as a doubling of the coefficient indexes. In bra-ket notation, one has $\langle em|\mathscr{L}_B|en\rangle = \delta_{m,2n}$, where we use the abbreviated notation $|en\rangle$ to stand for $\cos 2\pi nx$ or $\sin 2\pi nx$, as appropriate. Writing this out in matrix form, the Bernoulli operator has the distinctive form

$$\mathscr{L}_B = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \ldots & \\ \ldots & & & & & \end{array}\right] \tag{7.2}$$

with every other column consisting entirely of zeros. Clearly, this operator is quite singular, and has a large kernel.

The Koopman operator of the Bernoulli Map has the property of taking a function and making two copies of it. Working from it's definition, the Koopman operator is given by

$$\begin{aligned}[\mathscr{K}_B f](y) &= \int_0^1 \delta(x - b(y)) f(x)\, dx \\ &= \sum_{\forall x: x = b(y)} f(x) \\ &= f(2y)\theta(1-2y) + f(2y-1)\theta(2y-1)\end{aligned}$$

Here, $\theta(x)$ is the step function, identically zero for $x < 0$ and identically one for $x > 0$. Of course, $b(x)$ is the map 3.1. Acting on the Fourier series 7.1, one has that

$$[\mathscr{K}_B f](x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 4\pi nx + b_n \sin 4\pi nx$$

or, in Dirac notation, $\langle em | \mathscr{K}_B | en \rangle = \delta_{2m,n}$. In matrix form, it has the visual appearance of

$$\mathscr{K}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & \ldots & \\ \ldots & & & & & \end{bmatrix}$$

where every other row consists of zeros. It is obvious that $\mathscr{L}_B$ is literally the transpose: $\mathscr{K}_B = \mathscr{L}_B^T$ in this basis. Also quite evident is that, just as in the coordinate-space representation, one has that $\mathscr{L}_B \mathscr{K}_B = 1$ but $\mathscr{K}_B \mathscr{L}_B \neq 1$. The Koopman operator is the right-inverse for the Bernoulli operator, but it is not the left-inverse. In fact,

Theorem. *There does not exist a left inverse for $\mathscr{L}_B$.*

Proof. Assume that there exists an operator $\mathscr{M}$ such that $\mathscr{M}\mathscr{L}_B = 1$. Then, by associativity of operator multiplication, one has

$$\mathscr{M} = \mathscr{M} \cdot 1 = \mathscr{M} \cdot (\mathscr{L}_B \mathscr{K}_B) = (\mathscr{M}\mathscr{L}_B) \cdot \mathscr{K}_B = 1 \cdot \mathscr{K}_B = \mathscr{K}_B$$

which is clearly false. Thus, such an $\mathscr{M}$ does not exist. □

But, of course, $\mathscr{L}_B$ is singular, so such an inverse cannot exist.

The p-adic variants proceed in a similar fashion: acting on the basis 7.1, one has

$$[\mathscr{L}_p f](x) = a_0 + \sum_{n=1}^{\infty} a_{pn} \cos 2\pi nx + b_{pn} \sin 2\pi nx$$

and

$$[\mathscr{K}_p f](x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2p\pi nx + b_n \sin 2p\pi nx$$

Again, one has $\mathscr{L}_p \mathscr{K}_p = 1$ but $\mathscr{K}_p \mathscr{L}_p \neq 1$. The Koopman operators are also commutative and multiplicative: $\mathscr{K}_p \mathscr{K}_q = \mathscr{K}_q \mathscr{K}_p = \mathscr{K}_{pq}$.

It is instructive to verify that the Bernoulli polynomials are still eigenfunctions in this representation. For $n \neq 0$, one has

$$\int_0^1 B_1(x) \sin(2\pi nx)\, dx = \frac{-1}{\pi n}$$

and it is straightforward to "visually verify" that $\mathscr{L}_B B_1 = \frac{1}{2} B_1$, *i.e.* to take the above coefficients, and multiply them into the matrix 7.2. By working with the generator for the Bernoulli polynomials,

$$\frac{t\,e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}$$

one can immediately find, for $m \neq 0$, the Fourier components

$$\int_0^1 B_n(x)\,\cos(2\pi m x)\,dx = \begin{cases} 0 & \text{for n odd} \\ (-)^{1+n/2}\,n!/\,(2\pi m)^n & \text{for n even} \end{cases} \tag{7.3}$$

and

$$\int_0^1 B_n(x)\,\sin(2\pi m x)\,dx = \begin{cases} 0 & \text{for n even} \\ (-)^{(n+1)/2}\,n!/\,(2\pi m)^n & \text{for n odd} \end{cases} \tag{7.4}$$

Applying the Fourier-representation $\mathscr{L}_B$ to these to these vector components makes it immediately clear how the eigenvalue of $1/2^n$ is associated with the eigenvector B_n.

7.1. **The Kernel.** As noted earlier, the kernel of $\mathscr{L}_B$ consists of all the odd periodic functions $f(x) = -f\left(x+\frac{1}{2}\right)$ A basis for the kernel is given by $\cos 2\pi(2n+1)x$ and $\sin 2\pi(2n+1)x$ for positive integers n. The operator $P = \mathbb{I} - \mathscr{K}_B\mathscr{L}_B$ acts as a projection operator onto the kernel; we may easily verify that $P^2 = P$. Writing L_2 for the Hilbert space of square-integrable functions, the projection operator P splits this space into a direct sum $L_2 = B \oplus K$ with K the kernel. That is, every vector $v \in L_2$ may be written uniquely as the sum $v = b + k$ with $b = (\mathbb{I} - P)\,v \in B$ and $k = Pv \in K$. The subspaces B and K are both closed.

Although the space of polynomials embeds into L_2 and is likewise split, none of the elements of either B or K are polynomials. In particular, none of the Bernoulli polynomials B_n live entirely within B. This is perhaps surprising, given previous developments, but follows directly from eqns 7.3 and 7.4. Put in other words, $PB_n \neq 0$ and $(\mathbb{I} - P)\,B_n \neq B_n$. This seems surprising, because the B_n are eigenvectors $\mathscr{L}_B B_n = \lambda_n B_n$ and it is not uncommon that eigenvectors are orthogonal to the kernel. However, this need not be the case. Defining $k_n = PB_n$ and $b_n = (\mathbb{I} - P)\,B_n$ so that $B_n = b_n + k_n$, one then has that $\mathscr{L}_B k_n = 0$ while $\mathscr{L}_B b_n = \lambda_n\,(b_n + v_n)$. The Bernoulli operator acts in a skew fashion, twisting a portion of each vector b_n into the kernel K.

What's more, this skewness now 'explains' an earlier mystery. It was noted earlier that no polynomial lies in the kernel; this leads all too easily to the false assumption that every polynomial lies entirely within B. This mistaken assumption then leads to counter-intuitive results, such has how the sequence of the Bernoulli polynomials B_n can converge to something that is non-zero, yet also within the kernel. This in turn confuses the question of what the completion of the polynomials must be like. This can now be resolved.

So, for example, one has the Cauchy sequence $B_n(x) \to \cos 2\pi x$ (when the B_n are properly normalized). Both B and K are closed; that is, the limit of a Cauchy sequence in B (or K) is contained in B (respectively, K). Insofar as the sequence B_n seems to be in B, while clearly $\cos \in K$, the Cauchy sequence $B_n \to \cos$ appears to contradict the closure. But this is only appearance; the faulty assumption was that the B_n lie entirely within B. So, let $Q = \mathbb{I} - P$. Both Q and P are bounded operators, and thus continuous. Since Q is continuous, it follows that $QB_n \to Q\cos$, likewise $PB_n \to P\cos$. But $Q\cos = 0$, since $\cos \in K$, and $P\cos = \cos$. So one has, at last, that $QB_n \to 0$ while $PB_n \to \cos$. There is not one Cauchy sequence, but two, and both now converge as expected. The space of polynomials overlaps both B and K, as does it's completion.

The kernel for the p-adic operator $\mathscr{L}_p$ is spanned by $\cos 2\pi(pn+m)x$ and $\sin 2\pi(pn+m)x$ for positive integers n and $0<m<p$. That these lie in the kernel follows from the geometric sum $\sum_{k=0}^{p-1}\exp 2\pi ikn/p$ which vanishes whenever p does not divide n.

7.2. **The Wold decomposition.** The Koopman operator can be understood to be a shift operator[24], while the general theory of shift operators allows the Hilbert space on which a shift operator acts to be split into pieces, the Wold decomposition[25]. We review the decomposition here, as it applies in the present case.

Let $\mathscr{H}$ be a Hilbert space, and $\mathscr{B}(\mathscr{H})$ be the set of bounded linear operators on $\mathscr{H}$. An operator S in $\mathscr{B}(\mathscr{H})$ is a shift operator if S is an isometry, and if $\|S^{*n}f\|\to 0$ for all $f\in\mathscr{H}$. The Koopman operator fills the bill: clearly it leaves the length of vectors alone, as here, the norm $\|\cdot\|$ is just the usual norm on L_2. As we've seen, the Hermitian conjugate S^* is just the transpose, in this case, *i.e.* it is $\mathscr{L}_{\mathscr{B}}$. We've yet to show that the spectrum of $\mathscr{L}_B$ is less-than or equal to one; this is done in the coming sections. However, $\mathscr{L}_B$ does have one (unique) eigenvector associated with eigenvalue 1; if we exclude this from the Hilbert space $\mathscr{H}$, then, indeed, $\|\mathscr{L}_B^n f\|\to 0$ for all $f\in\mathscr{H}$. That is, we take $\mathscr{H}$ to be the subspace of L_2 with the constant term removed: *i.e.* all functions f with $a_0=0$ in equation 7.1.

The Wold decomposition may be given as follows:

Theorem. *(Wold decomposition) Given a shift operator $S\in\mathscr{B}(\mathscr{H})$ and $K=\ker S^*$, then $\mathscr{H}=\bigoplus_{n=0}^{\infty}S^nK$. Each $f\in\mathscr{H}$ may be written uniquely as*

$$f=\sum_{n=0}^{\infty}S^nk_n$$

where $k_n\in K$ is given by

$$k_n=PS^{*n}f$$

and $P=I-SS^*$ is the projection of $\mathscr{H}$ onto K. Furthermore, one has $\|f\|^2=\sum_{n=0}^{\infty}\|k_n\|^2$.

Proof. See [25, Chapter 1]. □

The above theorem holds in general for shift operators on Hilbert spaces. In our case, it is instructive to see how it plays out. We have, of course, $S=\mathscr{K}_B$ and $S^*=\mathscr{K}_B^T=\mathscr{L}_B$ and $P=\mathbb{I}-\mathscr{K}_B\mathscr{L}_B$ as before. The projection operator P projects a function f onto the kernel K spanned by $\cos 2\pi(2j+1)x$ and $\sin 2\pi(2j+1)x$ for positive integer j. The operator $P\mathscr{L}_B$ projects onto the space spanned by $\cos 4\pi(2j+1)x$ and $\sin 4\pi(2j+1)x$, and, in general, $P\mathscr{L}_B^n$ projects onto the space spanned by $\cos 2^{n+1}\pi(2j+1)x$ and $\sin 2^{n+1}\pi(2j+1)x$. That this is the entire space, and that the decomposition is unique, follows from the fact that any integer N can be written uniquely as $N=2^n(2j+1)$ for some integers n and j.

For the p-adic operator, similar remarks hold; defining $P_p=\mathbb{I}-\mathscr{K}_p\mathscr{L}_p$, one finds that $P_p\mathscr{L}_p^n$ projects onto the space spanned by $\cos\pi p^{n+1}(pj+m)x$ and $\sin\pi p^{n+1}(pj+m)x$ for positive integers j and $0<m<p$. Again, any integer N can be decomposed uniquely as $N=p^n(pj+m)$ for some integers n, j and $0<m<p$.

This same decomposition, but viewed from a different angle, recurs again, in a later section, in the construction and analysis of the 'fractal' spectrum of the Bernoulli operator. This time, the powers of 2 will be understood to be moves on the dyadic tree, with the Cantor set appearing as the limit of the dyadic tree, and the fractal self-symmetry following there-from. The take-away lesson here is, perhaps, that any system with a shift operator and/or an infinite Cartesian product of spaces will exhibit eigenfunctions with fractal-type symmetry. The self-symmetry arises purely from the interplay between the shift and the kernel.

7.3. **The Hurwitz zeta eigenfunctions.** The Bernoulli operator has a continuous spectrum, when considered in the space of square-integrable functions. The eigenvectors are quite degenerate, in that there is a countable infinity of eigenvectors associated with one eigenvalue. The degeneracy allows the eigenvectors to be combined in any of a variety of ways; different linear combinations have different properties, each interesting in its own way. In what follows, we will first present the Hurwitz zeta eigenfunctions. These generalize the Bernoulli polynomials, and, like the polynomials, are more or less "nice" and differentiable, at least away from the endpoints. An alternate linear combination will be presented in a later section: The Hurwitz zeta eigenfunctions can be re-summed to yield fractal curves, which are differentiable nowhere (well, strictly speaking, non-differentiable on the dyadic rationals). Each basis offers it's own insights.

Referring to the representation in eqn 7.2, it becomes visually clear that any vector with vector components $a_n = 1/n^s$ or $b_n = 1/n^s$ will be an eigenvector of $\mathscr{L}_B$ associated with the eigenvalue $\lambda = 1/2^s$. In coordinate space, one may write these eigenfunctions as

$$\beta(x;s) = 2\Gamma(s+1)\sum_{n=1}^{\infty}\frac{\exp(2\pi inx)}{(2\pi n)^s} \tag{7.5}$$

which transform as $\mathscr{L}_B\beta(x;s) = 2^{-s}\beta(x;s)$. This is not the only function associated with this eigenvalue: the eigenspace is infinite-dimensional. The Euler identity $1 = e^{2\pi im}$ implies that $2^{-s} = 2^{-(s+2\pi im/\ln 2)}$ and so

$$\beta_m(x;s) = \beta\left(x;s+\frac{2\pi mi}{\ln 2}\right)$$

is also an eigenvector: $\mathscr{L}_B\beta_m(x;s) = 2^{-s}\beta_m(x;s)$. Next, there is no reason to stick to only positive values of n in the summation. The space-reflected beta function is also an eigenvalue: that is, $\mathscr{L}_B\beta_m(1-x;s) = 2^{-s}\beta_m(1-x;s)$.

Up to the overall normalization, the function 7.5 is sometimes called the "periodic zeta function"[1, Sect. 12.7]. Although it superficially appears to be periodic in x, it is not actually so[32]: the value of $x = 1$ corresponds the branch point of the polylogarithm. The classical polylogarithm is defined as

$$\text{Li}_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}$$

and so

$$\beta(x;s) = \frac{2\Gamma(s+1)}{(2\pi)^s}\text{Li}_s\left(e^{2\pi ix}\right)$$

Although the polylogarithm can be analytically continued to all values of s and z, it has branch points at $z = 0$ and $z = 1$, and a non-trivial monodromy group[32].

This series recreates the Bernoulli polynomials for integer values of n, so for example, $\Re\beta(x;2) = B_2(x)$ and $\Im\beta(x;3) = B_3(x)$ and generally $\Re\left[(-i)^n\beta(x;n)\right] = -B_n(x)$. Equivalently, the Fourier series for the Bernoulli polynomials can be written as

$$\begin{aligned} B_n(x) &= -\Gamma(n+1)\sum_{k=1}^{\infty}\frac{\exp(2\pi ikx)+(-1)^n\exp(2\pi ik(1-x))}{(2\pi ik)^n} \\ &= \frac{-(-i)^n}{2}\left(\beta(x;n)+(-1)^n\beta(1-x;n)\right) \end{aligned}$$

See, for example [1, Thm. 12.19].

The periodic zeta function $\beta(x;s)$ is also an eigenfunction of the general p-adic operator, given in equation 3.3. That is, one has

$$\mathscr{L}_p \beta(x;s) = \frac{1}{p^s}\beta(x;s) \tag{7.6}$$

which may be demonstrated easily enough. As before, $\beta(1-x;s)$ is also an eignefunction; however, this time, the spanned space is different: additional eigenvectors are given by $\beta(x;s+2\pi im/\ln p)$: notice the $\ln p$ appearing in this expression, instead of the previous $\ln 2$.

These eigenfunctions are just linear combinations of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \tag{7.7}$$

and, in fact, the Hurwitz zeta itself is an eigenfunction of $\mathscr{L}_p$, with eigenvalue p^{s-1}. The Hurwitz zeta is related to β as

$$\beta(x;s) = \frac{is}{\sin \pi s}\left[e^{-i\pi s/2}\zeta(1-s,x) - e^{i\pi s/2}\zeta(1-s,1-x)\right]$$

This relationship is known as Jonquière's identity[20, Section 7.12.2][17] when expressed in terms of polylogarithms; its derivation is reviewed in Appendix E. It is straightforward to invert the above and solve for ζ; one gets

$$\zeta(1-s,x) = \frac{1}{2s}\left[e^{-i\pi s/2}\beta(x;s) + e^{i\pi s/2}\beta(1-x;s)\right]$$

thus proving the assertion that the Hurwitz zeta is an eigenfunction of $\mathscr{L}_p$, with eigenvalue p^{s-1}. A much easier proof is obtained by directly inserting eqn 7.7 into 3.3.

The beta functions are smooth (infinitely differentiable) for the open interval $0 < x < 1$, but posses an essential singularity at $x = 0$, and more precisely, depending on s, for all integer values of x. The singularity is of the form x^{s-1} as $x \to 0$. The eigenfunctions are square-integrable when $\Re s > 1$, although, depending on the value of s, the higher derivatives in general are not. The zeta functions have a simple pole at $s = 1$. The analytic properties are reviewed in Appendix F; a deeper, more complete picture of the analytic properties can be obtained by studying the polylogarithm[32]. The analytic properties of $\beta(x;s)$ for $\Re s \leq 1$ are non-trivial, due to the branch-point of the polylogarithm, and so naive manipulations can be dangerous to carry out. For practical work, the Hurwitz zeta function is better suited and easier to understand.

An important concluding remark is in order: these eigenfunctions are defined on the entire complex s-plane. The spectrum lying inside the unit disk $|\lambda| < 1$ corresponds to the right side of the complex plane: $\Re s > 0$ (for the Hurwitz zeta, $\Re s < 1$). Now, because these functions are well-defined, and even analytic, for the entire plane, we effectively have a family of eigenfunctions for all $|\lambda| > 1$, although these will not be integrable, due to singularities at $x = 0, 1$. This is notable because most general treatments of transfer operators, e.g. [5, 21] limit themselves to the space L_1 of integrable functions, and thus discover a spectrum on the unit disk $|\lambda| \leq 1$. The point here is that the spectrum need not stop at the unit disk; it can be larger, provided one expands the space of allowable functions.

These eigenfunctions are countably infinite degenerate at a given eigenvalue. Writing $s = \sigma + i\tau$ in terms of its real and imaginary components, any given eigenvalue takes the form $\lambda = p^{-s} = p^{-\sigma}\exp(-i\tau \ln p)$. Each such eigenvalue corresponds to a family of eigenvectors with $\tau' = \tau + 2\pi n/\ln p$ for $n \in \mathbb{Z}$. One may take linear combinations of eigenvectors in this family. Clearly, any linear combination of only a finite number of such eigenvectors will also be smooth, and have the same general analytic properties; this is not the case for certain infinite combinations.

In later sections, it will be seen the Bernoulli operator also has fractal eigenvectors. Some of these may be written as linear combinations of of the zeta eigenvectors, although clearly these require the linear combination of an infinite number of the zetas. This begs several questions: are all linear combinations of the zetas that require an infinite series fractal? Or are there infinite linear combinations that are smooth? Is the set of smooth solutions connected or disconnected? Are the connected components simply connected or not? Answers to these questions are not immediately apparent.

7.4. **Synthetic operators.** Starting from the definition 3.3, it is straight-forward to deduce that the p-adic operators commute: $\mathscr{L}_p\mathscr{L}_q = \mathscr{L}_q\mathscr{L}_p$ and that they are multiplicative: $\mathscr{L}_p\mathscr{L}_q = \mathscr{L}_{pq}$. This is re-expressed in equation 7.6, which just re-emphasizes these all posses the same eigenfunctions. Since they are commuting, one is free to take arbitrary sums; since they are multiplicative, an arbitrary product will always simplify. Thus, one may entertain the thought of various artificially-constructed or "synthetic" operators which share eigenfunctions but have novel eigenvalues. A few examples follow below.

Any Dirichlet series is straightforward: so, an operator with polylogarithm eigenvalues:

$$\left[\sum_{p=1}^{\infty} z^p \mathscr{L}_p\right] \beta(x;s) = \sum_{p=1}^{\infty} \frac{z^p}{p^s} \beta(x;s) = \mathrm{Li}_s(z)\beta(x;s) \tag{7.8}$$

There is a curious squaring:

$$\left[\sum_{p=1}^{\infty} e^{2\pi i y p} \mathscr{L}_p\right] \beta(x;s) = \sum_{p=1}^{\infty} \frac{e^{2\pi i y p}}{p^s} \beta(x;s) = \frac{(2\pi)^s}{2\Gamma(s+1)} \beta(y;s)\beta(x;s)$$

An operator built from the Mobius function μ:

$$\mathscr{M}_D = \sum_{p=1}^{\infty} \mu(p)\mathscr{L}_p \tag{7.9}$$

with an eigenvalue equation

$$\mathscr{M}_D\beta(x;s) = \sum_{p=1}^{\infty} \frac{\mu(p)}{p^s} \beta(x;s) = \frac{1}{\zeta(s)} \beta(x;s)$$

This "Mobius operator" $\mathscr{M}_D$ is explored further, below.

Because the operators are multiplicative, Euler products[1] may be taken. Thus, for example, setting $z = 1$ in eqn 7.8 gives $\mathrm{Li}_s(1) = \zeta(s)$, while factoring the summation into primes yields

$$\mathscr{L}_D = \sum_{p=1}^{\infty} \mathscr{L}_p = \prod_{p \text{ prime}} \frac{1}{1-\mathscr{L}_p} \tag{7.10}$$

The product is taken only over prime numbers p. The eigen-equation is

$$\mathscr{L}_D\beta(x;s) = \zeta(s)\beta(x;s)$$

which illustrates the curious situation that the vectors in the kernel of $\mathscr{L}_D$ are in correspondence with the zeroes of the Riemann zeta function. In what follows, we give this operator the name "divisor operator", as will be made clear in the next section.

The Koopman operator corresponding to $\mathscr{L}_D$ is

$$\mathscr{K}_D = \prod_{p \text{ prime}} \frac{\mathscr{K}_p - 1}{\mathscr{K}_p}$$

and again, $\mathscr{L}_D \mathscr{K}_D = 1$ but $\mathscr{K}_D \mathscr{L}_D \neq 1$. The projection onto the kernel is $P_D = 1 - \mathscr{K}_D \mathscr{L}_D$.

One may likewise consider any of the other typical Euler products encountered in number theory. For example, the operators

$$\sum_{p=1}^{\infty} f(p) \mathscr{L}_p$$

for $f(p)$ the (totally) mutiplicative totient function or the Liouville function, or for the incompletely multiplicative divisor functions.

With considerably more difficulty, one might envision operators constructed after a sequence of differential and integral moves; for example, something involving the Mellin transform. In principle, any such operator may be constructed. Denote $\beta(x;s)$ by the vector space basis vector e_s, and let e_s^* be its dual, so that $e_s^*(e_t) = \delta(s-t)$. Then given an arbitrary function $f(s)$, one may consider the operator constructed with the topological tensor product:

$$\mathscr{O} = \int_{\mathbb{C}} ds\, f(s)\, e_s \otimes e_s^*$$

The operator $\mathscr{O}$ has, by construction, eigenvectors $\beta(x;s)$ corresponding to eigenvalues $f(s)$. What makes the above different is that there is a certain air of concreteness to the p-adic transfer operators.

7.5. **The Divisor Operator.** The synthetic operator given by equation 7.10 is curious enough that it deserves explicit exploration. We give it the name "divisor operator", which can be understood by considering the operators $\mathscr{L}_p$ in the Fourier basis. In this basis, $\mathscr{L}_1 = \mathbb{I}$ is the identity operator, while $\mathscr{L}_2 = \mathscr{L}_B$ was previously shown in equation 7.2. To continue in this vein, $\mathscr{L}_3$ is explicitly given by

$$\mathscr{L}_3 = \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots \end{array}\right]$$

so that it has non-zero entries only along the elongated diagonal. If we number the rows and columns starting at zero, rather than one, it is easy to see that the only non-zero entries of $\mathscr{L}_p$ are $[\mathscr{L}_p]_{m,pm}$. That is, the general for for the matrix elements is given by

$$[\mathscr{L}_p]_{mk} = \delta_{pm,k}$$

Consider now the operator $\mathscr{L}_D = \sum_{p=1}^{\infty} \mathscr{L}_p$ written in this basis. It has the following form:

$$\mathscr{L}_D = \left[\begin{array}{cccccccccccc} \infty & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & & & & & & 1 \end{array}\right]$$

The zeroth column correspond to the coefficient a_0 for the Fourier representation 7.1; thus we conclude that $a_0 = 0$.

Ignoring the zeorth row and column, we see a clear pattern: the first row has all-ones in every position right of the diagonal. The second row has ones in every other column right of the diagonal. The third row has ones in every third column. That is, the rows indicate the divisibility of the column number: the matrix elements (ignoring row/column zero, again) are given by:

$$[\mathscr{L}_D]_{mk} = \begin{cases} 1 & \text{if } m|k \\ 0 & \text{otherwise} \end{cases} \tag{7.11}$$

where the notation $m|k$ means "m divides k". The reason for the name "divisor operator" is now clear: it comes from the divisor function of number theory, which is defined as[1]

$$d(k) = \sum_{m|k} 1$$

Indeed, summing over rows for a fixed column, we have that

$$d(n) = \sum_{k=1}^{\infty} [\mathscr{L}_D]_{kn}$$

The spectrum and eigenvectors of the divisor operator become clear when written in the matrix form: Clearly, one has

$$[\mathscr{L}_D] \begin{bmatrix} 1^{-s} \\ 2^{-s} \\ 3^{-s} \\ 4^{-s} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & & & & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} 1^{-s} \\ 2^{-s} \\ 3^{-s} \\ 4^{-s} \\ \vdots \end{bmatrix} = \zeta(s) \begin{bmatrix} 1^{-s} \\ 2^{-s} \\ 3^{-s} \\ 4^{-s} \\ \vdots \end{bmatrix}$$

or, in index-free notation:

$$\mathscr{L}_D v(s) = \zeta(s) v(s) \tag{7.12}$$

where $v(s)$ is the vector whose n'th component is $[v(s)]_n = a_n = n^{-s}$. Summing over the indexes of this vector simply generates a well-known Dirichlet series identity for the divisor function:

$$\sum_{k=1}^{\infty} [\mathscr{L}_D v(s)]_k = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [\mathscr{L}_D]_{kn} [v(s)]_n = \sum_{n=1}^{\infty} d(n) [v(s)]_n = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s) \sum_{k=1}^{\infty} [v(s)]_k = \zeta^2(s)$$

Thus, equation 7.12 can be thought of as a minor generalization of the above Dirichlet series.

How should the the kernel of the divisor operator be understood? Because we have that $\mathscr{L}_D \beta(x;s) = \zeta(s)\beta(x;s)$ and likewise for $\beta(1-x;s)$, we have that the kernel corresponds to the zeros of the Riemann zeta. For the regular zeros... finish me. The zeros on the critical strip are quite another matter.

7.6. **Moebius Inversion.** One may continue in this number-theoretic vein. Thus, the definition of the matrix elements given in 7.11 allows the following identity to be written. For any arithmetic series $f(n)$, one has that

$$\sum_{m|k} f(m) = \sum_{m=1}^{\infty} f(m) \left[\mathscr{L}_D\right]_{mk}$$

Inspired by the Dirichlet inverse of the Moebius function

$$\sum_{m|k} \mu(m) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

we may generalize by noting that, since $\mathscr{L}_D$ is upper-triangular (that is, $\left[\mathscr{L}_D\right]_{mk} = 0$ for $m > k$), the left-inverse is easily obtained, row by row, recursively:

$$\left[\mathscr{M}_{\mathscr{D}}\right]_{nk} = \begin{cases} 0 & \text{if } n > k \\ 1 & \text{if } n = k \\ -\sum_{m=n}^{k-1} \left[\mathscr{M}_D\right]_{nm} \left[\mathscr{L}_D\right]_{mk} & \text{if } n < k \end{cases}$$

so that $\mu(m) = \left[\mathscr{M}_D\right]_{1m}$ is the traditional Moebius function, while further rows are offset by one, so that

$$\mathscr{M}_D = \begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 1 & 0 & 0 \\ 0 & & & & \cdots & 0 & 1 & 0 \\ \vdots & & & & & \cdots & 0 & \ddots \end{bmatrix}$$

It's perhaps not obvious from casual examination of the above, but each row is just an elongated copy of the moebius function itself. That is,

$$\left[\mathscr{M}_{\mathscr{D}}\right]_{nk} = \begin{cases} 0 & \text{if } n > k \\ 1 & \text{if } n = k \\ \mu(k/n) & \text{if } n < k \text{ and } n|k \\ 0 & \text{otherwise} \end{cases}$$

and, by construction, this is the left inverse to the divisor function: $\mathscr{M}_D \mathscr{L}_D = \mathbb{I}$ or in index notation:

$$\sum_{k=1}^{\infty} \left[\mathscr{M}_{\mathscr{D}}\right]_{nk} \left[\mathscr{L}_D\right]_{km} = \delta_{nm}$$

In fact, $\mathscr{M}_D$ is also the right-sided inverse: $\mathscr{L}_D \mathscr{M}_D = \mathbb{I}$, which is easy to verify. Thus, in the Fourier representation, we may correctly write $\mathscr{M}_D = \mathscr{L}_D^{-1}$ and $\mathscr{L}_D = \mathscr{M}_D^{-1}$ although this is a paradoxial result: both operators have a non-trivial kernel! We return to this below.

It is not hard to see that the Mobius operator $\mathscr{M}_D$ constructed here is identical to the synthetic operator constructed in equation 7.9: $\mathscr{M}_D = \sum_{p=1}^{\infty} \mu(p) \mathscr{L}_p$. Thus, it follows, through a variety of different arguments, whether from equation 7.12 or by general appeal to operator theory, that the eigenvalues of $\mathscr{M}_D$ are $1/\zeta(s)$ and the eiegenvectors are $v(s)$ with $[v(s)]_n = a_n = n^{-s}$ are before:

$$\mathscr{M}_D v(s) = \frac{1}{\zeta(s)} v(s)$$

The first row of the above operator equation is none other than the well-known Dirichlet series identity

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

and in fact, every row of the operator equation simply repeats this form. Going over to the Fourier decomposition 7.1, the position-space eigenvectors are again given by equation 7.5 (keeping in mind, as before, that we are ignoring the zero'th column/row, or, equivalently, that $a_0 = 0$), and so one has that

$$[\mathscr{M}_D \beta(s)](x) = \frac{\beta(x;s)}{\zeta(s)}$$

with the same identity holding for $\beta(1-x;s)$. Because the zeta function has only one pole, at $s = 1$, this implies that the Moebius operator has a two-dimensional kernel. It is spanned by the generator

$$\beta(x;1) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{\pi n}$$

and by $\beta(1-x;1)$. Unfortunately, these sums do a dis-service: really, the difference just gives the Bernoulli polynomial. That is, $\frac{i}{2}(\beta(x;1) - \beta(1-x;1)) = B_1(x) = x - \frac{1}{2}$, while the sum $\frac{1}{2}(\beta(x;1) + \beta(1-x;1)) = \frac{1}{\pi}\mathrm{Cl}_1(2\pi x)$ where $\mathrm{Cl}_1(\theta)$ is the Clausen function of order 1. It is purely real, very smooth, having a U-shape, with singularities at $x = 0$ and 1.

7.6.1. *Invertability and the Riemann Hypothesis.* What is the correct way of understanding the operator equations $\mathscr{M}_D = \mathscr{L}_D^{-1}$ and $\mathscr{L}_D = \mathscr{M}_D^{-1}$ given that both of these operators appear to have a non-trivial kernel? The resolution of the paradox requires paying careful attention to the spaces on which they act. From the point of view of classical number theory, the operators are restricted to act only on the Banach space $l_1 = \{(a_1, a_2, a_3, \cdots) : \sum_{n=1}^{\infty} |a_n| < \infty\}$ of absolutely summable series. None of the singular solutions belong to this space. Thus, the vector $a_n = 1/n$ lies in the kernel of $\mathscr{M}_D$ but it is clearly not l_1-summable. That's OK, because Fourier analysis asks for l_2-summability, and this vector is clearly summable with the l_2 norm.

The situation for the divisor operator $\mathscr{L}_D$ is a bit more confusing. Its kernel corresponds to the zeros of the Riemann zeta function. The vector $a_n = n^{-s}$ is again not summable in the l_1-norm when $\Re s \leq \frac{1}{2}$, which is the region where all of the Riemann zeros are conjectured to lie.

In that sense, we have a simple statement that is equivalent to the Riemann hypothesis. Assume that the Riemann zeta had a zero in the critical strip that did not lie on the critical line. By symmetry, this zero is doubled. That is, write $s = \frac{1}{2} + \eta + i\tau$ for η, τ real. If there is a zero with $\eta > 0$, then there is a second zero at $\frac{1}{2} - \eta + i\tau$. The zero with the positive η corresponds to a series that is l_2-summable. That is, we have that $\sum_{n=1}^{\infty} \left| n^{\frac{1}{2}+\eta+i\tau} \right|^2 < \infty$ whenever $\eta > 0$. If the Riemann hypothesis holds, then we may conclude that the divisor operator $\mathscr{L}_D$ is not singular, and thus is invertible, on the space of Fourier series (i.e. series that are l_2-summable). We have already proven that $\mathscr{L}_D$ is invertible on the Banach space

l_1. If we could prove that it was invertible on the Banach space l_2, then we would have a proof of the Riemann hypothesis.

Assuming RH, then we see that the operator $\mathscr{L}_D$ is singular only when the space of functions on which it acts can be properly analytically continued into the regions where $\Re s \leq \frac{1}{2}$.

7.7. **More Number Theory.** Further well-known identities from Dirichlet convolution follow in a similar fashion. Let $1 = (1,1,1,\cdots)$ be the row vector of all-ones; let $\varepsilon = (1,0,0,\cdots)$ be the row vector with one in the first position, and then all zeros. Then clearly, $1\mathscr{M}_D = \varepsilon$ is the operator equation analog of the Dirichlet convolution $1 * \mu = \varepsilon$. The Moebius inversion formula, with states that $g = f * 1$ if and only if $f = g * \mu$ has a clear operator analog: $\mathscr{G} = \mathscr{F}\mathscr{L}_D$ if and only if $\mathscr{F} = \mathscr{G}\mathscr{M}_D$ which follows trivially from the identity $\mathscr{L}_D\mathscr{M}_D = \mathbb{I}$ which was obtained above. As before, this Dirichlet convolution identity holds only on the space of l_1-summable series, and it holds on the space of l_2-summable series, if RH holds. Clearly, the identities cannot hold when the space of functions is expanded by analytic continuation, as the kernels of $\mathscr{F}$ and $\mathscr{G}$ would be, by construction, become tainted by the zeros of $\mathscr{M}_D$ and $\mathscr{L}_D$.

By analogy, the generalized divisor operator corresponding to the sigma function is then

$$\mathscr{L}_\Sigma(z) = \left[\begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & & & & \\ 0 & 2^z & 0 & 2^z & 0 & 2^z & 0 & 2^z & 0 & \cdots & \\ 0 & 0 & 3^z & 0 & 0 & 3^z & 0 & 0 & 3^z & 0 & \cdots \\ 0 & 0 & 0 & 4^z & 0 & 0 & 0 & 4^z & 0 & 0 & \cdots \\ & \cdots & 0 & 0 & 5^z & 0 & 0 & 0 & 5^z & 0 & \cdots \\ & & & & & 6^z & & & & & \ddots \end{array}\right]$$

where one has, as before, the sum-of-divisors function:

$$\sigma_z(n) = \sum_{m|n} m^z = \sum_{k=1}^{\infty} \left[\mathscr{L}_\Sigma(z)\right]_{kn}$$

Writing the diagonal operator $\left[\mathscr{Z}(z)\right]_{mk} = m^z \delta_{mk}$ as

$$\mathscr{Z}(z) = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \\ 0 & 2^z & 0 & 0 & 0 & \\ 0 & 0 & 3^z & 0 & 0 & \cdots \\ 0 & 0 & 0 & 4^z & 0 & \cdots \\ \vdots & & & & & \ddots \end{array}\right]$$

one obviously has, in this basis, that $\mathscr{L}_\Sigma(z) = \mathscr{Z}(z)\mathscr{L}_D$. This is the operator analog of the Dirichlet convolution $\sigma_z = \text{Id}_z * 1$. From the previous secion, we can deduce that $\mathscr{Z}(z) = \mathscr{L}_\Sigma(z)\mathscr{M}_D$ with is the operator analog of the convolution $\text{Id}_z = \sigma_z * \mu$.

The Euler totient function φ is given by the convolution $\varphi = \text{Id} * \mu$ which suggests that the correct operator analog is $\mathscr{P} = \mathscr{Z}(1)\mathscr{M}_D$ which has the following form:

$$\mathscr{P}=\left[\begin{array}{cccccccc} 1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 6 & 0 & 0 \\ 0 & & & & \cdots & 0 & 7 & 0 \\ \vdots & & & & & \cdots & 0 & \ddots \end{array}\right]$$

It is straight-forward to verify that the column sums give the Euler totient function: viz:

$$\varphi(n)=\sum_{k=1}^{\infty}[\mathscr{P}]_{kn}$$

Likewise, Jordan's totient function, given by the convolution $J_z=\mathrm{Id}_z*\mu$ has an operator analog that is given by $\mathscr{J}(z)=\mathscr{Z}(z)\mathscr{M}_D$. The Jordan totient function itself arises as the column sums, as before:

$$J_z(n)=\sum_{k=1}^{\infty}[\mathscr{P}(z)]_{kn}$$

One may continue in a similar vein, and define the operator analogs for the Liouville function, the von Mangoldt function, and so on. By means of the Fourier basis 7.1 and the Bernoulli operator in this basis 7.2, each of these operator analogs are "synthetic" operators that can be constructed from the (p-adic) Bernoulli operators.

There are multiple tasks which lie ahead:

- Explore the spectrum of the various operators defined above. As should be clear from the matrix form of $\mathscr{L}_\Sigma(z)$ above, the diagonal values, and thus the spectrum, is well-behaved only when $z<0$. Some form of analytic continuation is needed to move to the region $z>0$. Similar remarks apply for the totient function.
- Explore the analog to the Lambert series. This should be concistent with the other developments: its well-known that the Lambert series plays a role in the Eisenstein series and in the Jacobi theta functions. It is also well-known that the modular group $PSL(2,\mathbb{Z})$ describes the symmetries seen in these series. What is not well-known is the explicit relationship to the dyadic monoid and the Cantor set (each of these are effectively "half" of the modular group, so the traces of that relationship should survive.)

7.8. **Multiplication theorem.** The well-known multiplication theorem[35] of the special functions follows from the general case for multiplicative functions, given below. A hint of the multiplication theorem is already visible in equation 3.3. Here, we give it full expression.

Given some series $\{f_n\}$, consider the synthetic operator

$$\mathscr{F}=\sum_{n=1}^{\infty}f_n\mathscr{L}_n \tag{7.13}$$

Working in the Fourier representation, the operators $\mathscr{L}_n$ have the matrix elements

$$[\mathscr{L}_n]_{mk}=\delta_{mn,k}$$

Acting on vector $c = \{c_n\}$, the operator $\mathscr{F}$ gives

$$[\mathscr{F}c]_m = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} f_n \delta_{mn,k} c_k = \sum_{n=1}^{\infty} f_n c_{mn}$$

If the series $c = \{c_n\}$ is totally multiplicative, then, by definition of multiplicativity, one has that $c_{mn} = c_m c_n$ for all m,n, and so

$$[\mathscr{F}c]_m = c_m \sum_{n=1}^{\infty} f_n c_n$$

Writing this as an operator equation, we have

$$\mathscr{F}c = (f \cdot c)c$$

where the eigenvalue is the dot product $f \cdot c$ and c is the eigenvector. What have we done here? For any series $\{f_n\}$, we have constructed an operator $\mathscr{F}$ that has, as eigenvectors, any totally multiplicative series!

As one special case, we immeditately have that

$$\mathscr{F}\beta(x;s) = \lambda\beta(x;s)$$

with the eigenvalue being the Dirichlet series $\lambda = \sum_{n=1}^{\infty} f_n n^{-s}$. But of course: the synthetic operators 7.13 are just linear combinations, and so the eigenvectors are unchanged.

7.8.1. *The Multiplication theorem.* As another a special case, any totally multiplicative series is an eigenvector of the Bernoulli operator. That is:

$$\mathscr{L}_p c = c_p c$$

By analogy to the beta functions defined in7.5, the explicit functional form may be given:

$$\gamma(x) = \sum_{n=1}^{\infty} c_n \exp(2\pi i n x)$$

Then $\gamma(x)$ together with $\gamma(1-x)$ satisfy the Ruelle-Frobenius-Perron operator or transfer operator of the Bernoulli map, as given in equation 3.3. That is, we have, by construction, that

$$[\mathscr{L}_p\gamma](x) = \frac{1}{p}\sum_{k=0}^{p-1}\gamma\left(\frac{x+k}{p}\right) = c_p\gamma(x) \tag{7.14}$$

This last is recognizable as the "multiplication theorem" of the special functions[35], thus giving a general way of constructing functions that obey this theorem. By letting $c_n = n^{-s}$, and applying various limits, one obtains each of the well-known multiplication theorems for the special functions. The periodic zeta function, the Hurwitz zeta function, and the Bernoulli polynomials were previously covered above. The polylogarithm follows easily, with little effort. The polygamma function presents a different and more interesting case.

For those multiplicative series that are not summable, some form of analytic continuation must be found, in order to give meaningful results.

7.8.2. *Summability.* The intersting question, from the point of view of having a functional analysis, is to answer the question “which totally multiplicative series belong to l_1 or l_2”? Of course, the series n^{-s} for $\Re s > 1$ (or $\Re s > \frac{1}{2}$ for l_2) satisfy this requirement. Are there other series? The usual suspects: the totient function, the divisor function, the Liouville function, the von Mangoldt function are not actually totally multiplicative; they are multiplicative only when thier arguments are coprime. The only well-known completely multiplicative functions are the Liouville function, and the Dirichlet characters. The Liouville function $\lambda(n)$ is not l_1-summable, but the function $\lambda(n)n^{-s}$, for suitable s, is (the product of any two multiplicative functions is multiplicative). Likewise, the Dirichlet charaters $\chi(n)$ themselves are not summable, but the functions $\chi(n)n^{-s}$ are.

There is another noteworthy possibility. If $a(n)$ is a totally additive function, so that $a(mn) = a(m) + a(n)$, then $c_n = z^{a(n)}$ is totally mutiplicative, and is obviously summable for $|z| < 1$. There are two well-known totally additive functions: $\Omega(n)$, the total number of prime factors of n counted with multiplicity, and $\nu_p(n)$, the number of times that the prime factor p occurs in the factorization of n.

Since the product of any two multiplicative functions is multiplicative, we can create arbitrary chimeras, such as $c_n = z^{a(n)}n^{-s}$. Each of these gives a distinct set of eigenvector equations. What is not clear is if, or how, any of these might be linear combinations of each other. That is, fix a constant λ, so that $\mathscr{L}_p c = \lambda c$. What is the space spanned by all such c's? Can an orthogonal basis be given? What are the smallest sets of totally mutiplicative functions that span this space? None of these answers seem obvious. There is a very rich collection of eigenfunctions of the Bernoulli operator, with no straightforward way of characterizing them.

7.9. **Hardy Classes.** The p-adic Bernoulli operator is a (conjugate of a) shift operator. Given that equation 7.6 indicated that these all commute, one has a family of commuting shift operators. An operator that commutes with a shift operator may be called “analytic”, in a certain precise sense, and, given appropriate properties, can often be further factored into Hardy classes[25], in analogy to the factorization into inner and outer functions of ordinary Hardy space.

XXX Do this.

8. Lattice model topology

It is well-known that one-dimensional lattice models, and more generally, subshifts of finite type, can provide useful tools and insights for the study of one-dimensional iterated maps[23]. This section recaps the use of a one-dimensional spin lattice for the construction of a class of Bernoulli operator eigenfunctions. In essence, the lattice models recapture the Wold decomposition of section 7.2, but using a more articulated, richer vocabulary. In particular, the shift operator gets a more concrete interpretation as a translation on a lattice.

Consider a one-sided lattice of points, labeled by the positive integers, that is, by $n \in \mathbb{N}$. At each lattice location, a variable σ_n, called the “spin” (a name deriving from physics), can take one of p values, so that $\sigma_n \in \{0, 1, 2, \cdots, p-1\}$. The space of states of the lattice is then $\Omega = p \times p \times \cdots = p^\omega$ where we write $p \equiv \{0, 1, 2, \cdots, p-1\}$, letting the number stand for the set. A given lattice configuration $\sigma = (\sigma_1, \sigma_2, \cdots) \in \Omega$ can be taken to be either a p-adic number, or as a real number x, with the Cantor mapping $x : \Omega \to [0, 1]$ from the set of all lattice configurations Ω to the unit interval $[0, 1]$ given by

$$x(\sigma) = \sum_{n=1}^{\infty} \sigma_n p^{-n} \tag{8.1}$$

This mapping allows every possible spin lattice configuration to be mapped into the unit interval $[0,1]$, although not vice-versa: the natural topology on the spin lattice is finer than the natural topology on the reals, in that, for example, $\sigma = (0,1,1,\cdots)$ and $(1,0,0,\cdots)$ are inequivalent lattice configurations, whereas the binary numbers $0.0111\ldots$ and $0.1000\ldots$ both represent the rational 1/2. Only certain rationals are double-pointed like this, whereas all irrationals correspond to a unique lattice configuration. For $p = 2$, the Stone representation theorem applies.

Any Cartesian product of spaces has a natural topology, called the "product topology"; the basis of the topology are referred to as "cylinder sets"[18, 23]. Since the one-dimensional lattice is the Cartesian product of an infinite number of finite sets, the product topology applies. In fact, the product topology will prove very nice to work with; it is a reasonably coarse topology, yet, it is compatible with the natural topology on the reals. One particularly strong utility of the product topology is that it enables the definition and discussion of measurable (integrable) functions that are discontinuous at all rational numbers; by contrast, such constructions are extremely awkward when using Fourier series, and, of course, impossible in the language of analysis. We'll construct many such functions here; in what follows, the product topology will be made great use of.

The product topology allows a set of fractal eigenfunctions to be constructed for the Bernoulli operator. One begins by noting that the left-shift operator τ on the lattice, defined by

$$\tau(\sigma) = \tau((\sigma_1, \sigma_2, \cdots)) = (\sigma_2, \sigma_3, \cdots)$$

corresponds exactly to the Bernoulli map:

$$x(\tau(\sigma)) = 2x(\sigma) - \lfloor 2x(\sigma) \rfloor$$

where we've written $x(\sigma)$ instead of x to help remind that x should be thought of as a map, not a real number. For the remainder of this section, this distinction will rarely be made again, although every occurrence of x should be implicitly understood to be the map $x : \Omega \to [0,1]$.

Consider now some arbitrary function $V : \Omega \to F$, where F may be taken to be the field $\mathbb{R}$, or $\mathbb{C}$, or some arbitrary general field. Then, given some $\lambda \in F$, one may construct the sum

$$H(\sigma) = \sum_{n=1}^{\infty} \lambda^n V(\tau^n \sigma) \tag{8.2}$$

where $\tau^n \sigma = (\tau \circ \tau \circ \cdots \circ \tau)(\sigma)$ is simply the n-fold iteration of the shift operator. In physics, the function H is usually called the translation-invariant Hamiltonian for the system, while V is the interaction potential. For a one-sided lattice, H is not strictly translation invariant, as

$$H(\tau(\sigma)) = \sum_{n=1}^{\infty} \lambda^n V(\tau^{n+1} \sigma) = \frac{H(\sigma) - V(\sigma)}{\lambda} \tag{8.3}$$

so that H is "almost" an eigenvector of τ, with eigenvalue $1/\lambda$. Almost, because the one-sided lattice introduces a correction V/λ. For the bi-infinite, two-sided lattice, a truly translation-invariant Hamiltonian can be defined. The two-sided lattice model corresponds to the Bakers map, and is treated in greater detail in section 9.

If τ is the left-shift operator, then the analog of the transfer operator can be roughly taken to be the right-shift operator. Since the lattice is one-sided, there exist p inequivalent right-shift operators S_k:

$$S_k(\sigma_1, \sigma_2, \cdots) = (k, \sigma_1, \sigma_2, \cdots) \tag{8.4}$$

These right-shift operators are inverses to the left shift operator τ only on one side, in that $\tau \circ S_k$ is the identity, but $S_k \circ \tau$ is not. Clearly, τ has no left-inverse, as τ throws away data in its action. For the $p = 2$ case, the shift operators act on the Cantor mapping as

$$x(S_0(\sigma)) = \frac{x(\sigma)}{2}$$
$$x(S_1(\sigma)) = \frac{1+x(\sigma)}{2}$$

and so one may immediately recognize how to pose the Bernoulli operator on the lattice model. For some arbitrary $f : \Omega \to F$ one defines

$$[\mathscr{L}_B f](\sigma) = \frac{1}{2}[f(S_0(\sigma)) + f(S_1(\sigma))]$$

For the general p-adic case, this generalizes trivially:

Definition. The p-adic Bernoulli operator acting on the space of functionals of lattice configurations is given by

$$[\mathscr{L}_p f](\sigma) = \frac{1}{p}\sum_{k=0}^{p-1} f(S_k(\sigma)) \tag{8.5}$$

where S_k is the k'th right-shift operator.

The action of $\mathscr{L}_p$ on the Hamiltonian H is takes the form

$$\mathscr{L}_p H = \mathscr{L}_p V + \lambda H \tag{8.6}$$

This is easily shown by direct substitution.

The critical result here is that H is an eigenvector provided that $\mathscr{L}_p V = 0$. Curiously, this implies that functions in the kernel of $\mathscr{L}_B$ can be used to construct eigenfunctions of $\mathscr{L}_B$. The above construction is not limited the the Bernoulli operator, but clearly is a general result: it is essentially the Wold decomposition of section 7.2, but restated in a a more general form: rather than requiring a Hilbert space, it holds for arbitrary Cartesian products over arbitrary spaces, provided only that the summation in eqn 8.2 is convergent (and/or meets any other criteria for the space being worked).

One may construct a large number of general eigenfunctions, provided that on can find non-trivial objects in the kernel, and provided that one is working in a topology where the sum of equation 8.2 converges.

8.1. **Fractal eigenfunctions.** The above derivation of the eigenfunctions assumed a lattice model (Cartesian product) topology for the unit interval. Using equation 8.1 to regain the natural topology on the real numbers, one may use the construction to rediscover the full set of eigenfunctions of the Bernoulli operator. This is done explicitly in this section.

From equation 6.1, it was seen that the kernel of the Bernoulli operator consists of the odd harmonics of sine and cosine:

$$\mathscr{L}_B \sin 2\pi(2k+1)x = 0$$
$$\mathscr{L}_B \cos 2\pi(2k+1)x = 0$$

Here, k is a non-negative integer parametrizing the kernel; however, it will be more convenient to write $\exp 2\pi i(2k+1)x$ and let k rang e over all integers, positive and negative. A simple linear combination over positive and negative k will regain the real form. Inserting these into equation 8.2, one has

$$H(x)=\sum_{n=0}^{\infty}\lambda^{n}\exp\left(2\pi\left(2k+1\right)\tau^{n}x\right)$$

which is an eigenfunction: $\mathscr{L}_B H=\lambda H$. Because exp is periodic, we may replace $\tau^n(x)$ by $2^n x$. Switching notation, one has a collection of eigenfunctions

$$\phi_{\lambda,k}^{B}(x)=\sum_{n=0}^{\infty}\lambda^{n}\exp\left(2\pi i\,2^{n}\left(2k+1\right)x\right) \tag{8.7}$$

for any $k\in\mathbb{Z}$. This derivation allows $\lambda\in F$ for any field F; taking F to be the complex numbers, the sum converges absolutely for the disk $|\lambda|<1$ and conditionally for $|\lambda|=1$, depending on whether λ is a root of unity or not.

The superscript B denotes that these are eigenfunctions for the $p=2$ case of the Bernoulli operator. For the general case, one builds the eigenfunctions out of the shift states

$$v_{n,k,r}^{(p)}(x)=\exp 2\pi ix\left(pk+r\right)p^{n}$$

for $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$ and r the "remainder", $0<r<p$.

Theorem 1. *The functions* $v_{n,k,r}^{(p)}(x)$ *are shift states of* $\mathscr{L}_p$*, that is,*

$$\left[\mathscr{L}_p v_{n,k,r}^{(p)}\right](x)=v_{n-1,k,r}^{(p)}(x) \tag{8.8}$$

for $n\neq 0$ *and* $\left[\mathscr{L}_p v_{0,k,r}^{(p)}\right](x)=0$ *for the* $n=0$ *case.*

Proof. This is obtained by simple substitution:

$$\begin{aligned}\left[\mathscr{L}_p v_{n,k,r}^{(p)}\right](x) &= \frac{1}{p}\sum_{m=0}^{p-1}v_{n,k,r}^{(p)}\left(\frac{x+m}{p}\right)\\ &= \frac{1}{p}\exp\left[2\pi ix\left(pk+r\right)p^{n-1}\right]\sum_{m=0}^{p-1}\exp\left[2\pi i\frac{m}{p}(pk+r)p^{n}\right]\\ &= v_{n-1,k,r}^{(p)}(x)\end{aligned}$$

The sum in the middle line above is a Gauss sum[1], and may be shown to equal p when $n\neq 0$ and equal to zero otherwise. □

The above relationship is reminiscent of of the creation and annihilation ladder operators of quantum mechanics. This correspondence can be made stronger, and is explored in a later section. The p-adic shift states of equation 8.8 can be trivially used to construct the eigenstates of the p-adic Bernoulli operator. These are given by

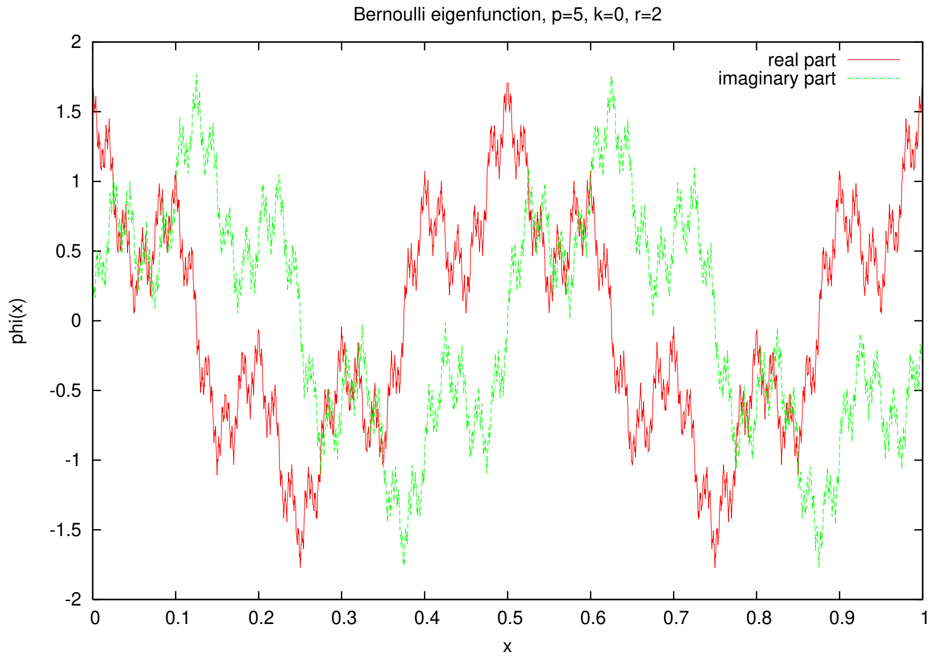
$$\phi_{\lambda,k,r}^{(p)}(x)=\sum_{n=0}^{\infty}\lambda^{n}\exp\left(2\pi ix\;\left(pk+r\right)p^{n}\right)$$

which clearly obey

$$\mathscr{L}_p\phi_{\lambda,k,r}^{(p)}=\lambda\phi_{\lambda,k,r}^{(p)}$$

These eigenfunctions are highly degenerate, and span an infinite-dimensional space labeled by k and r. Any linear combination of these is also an eigenvector with the same eigenvalue. These clearly form a continuous spectrum for the Bernoulli operator, although, strictly speaking, this spectrum is defined only on the unit disk $|\lambda|<1$. This is in contrast to

FIGURE 8.1. Fractal Eigenfunction



This figure shows the real and imaginary parts of the eigenfunction $\phi^{(p)}_{\lambda,k,r}(x) = \phi^{(5)}_{0.45,0,2}(x)$. The non-differentiability is clear visible. This function would be differentiable only for $\lambda < 0.2$; in that case, visually, the spikes would be completely smoothed away. The twice-repeated sine-wave nature is entirely due to $k = 2$.

---

the zeta eigenfunctions, which are defined for the entire complex λ-plane (although these are not integrable when $|\lambda| > 1$). This implies that the construction of eigenfunctions by means of lattice shifts is somehow innately limited: it allows some, but not all possible eigenfunctions to be found. The true nature of this obstruction is unclear. The proximal cause of the obstruction is that $\sum_n \lambda^n = 1/(1-\lambda)$ has a pole at $\lambda = 1$, thus preventing the convergence of the analytic series beyond that point. However, in the more global view, one has an operator acting on an infinite-dimensional topological vector space; whether the obstruction is due to the operator or the structure of the space itself is unclear.

8.2. **Analytic properties.** Unlike the zeta function eigenvectors, these eigenvectors are not infinitely differentiable. Taking the first derivative, one has

$$\frac{d}{dx}\phi^{(p)}_{\lambda,k,r}(x) = 2\pi i\,(pk+r)\sum_{n=0}^{\infty} p^n \lambda^n \exp\left(2\pi i x\ (pk+r)\,p^n\right)$$

and so the right hand side converges absolutely only if $|\lambda| < 1/p$. For $|\lambda| > 1/p$, the eigenfunction is continuous but not differentiable. This is visualized in figure 8.1. Second derivatives may be taken only if $|\lambda| < 1/p^2$; more generally, n'th derivatives may be taken only when $|\lambda| < 1/p^n$.

Clearly, any linear combination of only a finite number of these fractal eigenfunctions will posses similar analytic properties. However, there is an infinite series combination of these that does reproduce the zeta function eigenfunctions. An explicit relationship between these two is developed in the next section.

8.3. **The Fractal Spectrum.** The fractal spectrum can be re-expressed in terms of the zeta eigenfunctions, and vice-versa. That is, the one set of eigenfunctions can be expressed as linear combinations of the other. This section gives the explicit linear relations connecting the one to the other.

The establishment of this result is the observation that every integer has a unique factorization in the form of $(pk+r)p^n$, with k,r and n uniquely determined when p is fixed. This leads to a summation identity: for a fixed integer p, and some arbitrary summable function $f(n)$, one has

$$\sum_{n=0}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \sum_{m=0}^{\infty} f\left((pk+r)p^m\right)$$

The summand of the triple sum on the right is precisely of the form appearing in the fractal eigenfunctions, whereas the left-hand side is of the simpler form of the zeta eigenfunctions. The desired linear relationship is obtained merely by matching terms in the series. That is, one sets $f(n) = 2\Gamma(s+1)\,(2\pi n)^{-s} \exp(2\pi ixn)$ to immediately obtain

$$\begin{aligned} \beta(x;s) &= \frac{2\Gamma(s+1)}{(2\pi)^s} \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \frac{1}{(pk+r)^s} \sum_{m=0}^{\infty} \frac{\exp 2\pi ix(pk+r)p^m}{p^{ms}} \\ &= \frac{2\Gamma(s+1)}{(2\pi)^s} \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} \frac{1}{(pk+r)^s}\, \phi^{(p)}_{\lambda,k,r}(x) \end{aligned}$$

where one identifies $\lambda = p^{-s}$.

The full set of zeta eigenfunctions, accounting for degeneracy, is given by

$$\beta_n(x;s) \equiv \beta(x; s+2\pi ni/\ln 2)$$

with $n \in \mathbb{Z}$. These share the same eigenvalue: $\mathscr{L}_p \beta_n = p^{-s}\beta_n$. The relevant values of s should be restricted to a principal domain $-\pi < \Im s \ln 2 = \arg z < \pi$. Repeating the above manipulations, the goal is to write the zetas as a linear combination of the fractal eigenfunctions, so:

$$\beta_n(x;s) = \sum_{k=0}^{\infty} \sum_{r=1}^{p-1} F_{nkr}\, \phi^{(p)}_{\lambda,k,r}(x)$$

These matrix elements are

$$F_{nkr} = 2\Gamma\left(s+1+\frac{2\pi ni}{\ln 2}\right) (2\pi(pk+r))^{-s} \exp\left[-2n\pi i \frac{\ln \pi(pk+r)}{\ln 2}\right]$$

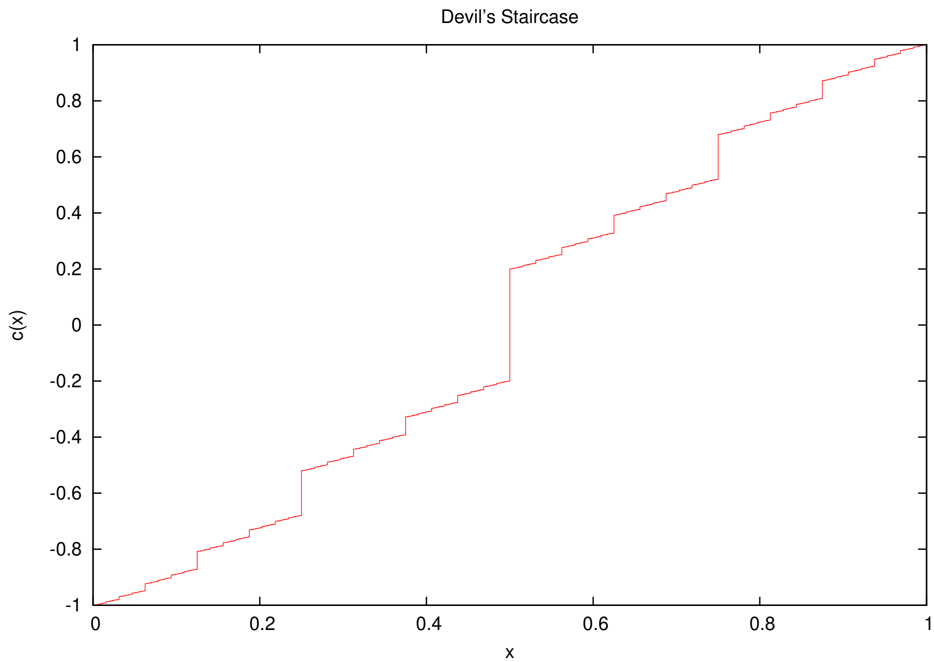
Presumably F is invertible; the $\phi^{(p)}_{\lambda,k,r}$ should be expressible in terms of the $\beta_n(x;s)$.

The analytic situation is curious: the zeta eigenfunctions are well-defined for $|\lambda| > 1$, even as the fractal eigenfunctions are not. The change-of-basis summations clearly appear to be divergent as $|\lambda|$ approaches 1. It would appear that the zeta eigenfunctions span a larger space than the fractal eigenfunctions.

8.4. **Discontinuous eigenfunctions.** Some of the eigenfunctions are not continuous in the natural topology of the reals. Thus, for example, given the expansion in binary digits $b_n \in \{0,1\}$ of a real number $0 \le x \le 1$,

$$x = \sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}}$$

FIGURE 8.2. Devil's staircase



This figure shows a graph of the eigenfunction $c_{\lambda}(x)$ for $\lambda = 0.4$. As can be clearly seen, it takes on values in the Cantor set.

Then the Cantor or "Devil's staircase" function

$$c_{\lambda}(x) = \lambda(\lambda - 1)\sum_{n=0}^{\infty}(-1)^{b_n}\lambda^n$$

is easily shown to be an eigenfunction of $\mathscr{L}_B$ with eigenvalue λ. This function is clearly discontinuous for all λ except $\lambda = 1/2$, where $c_{\lambda}(x) = B_1(x) = x - 1/2$. The Devil's staircase function is shown in 8.4.

This eigenfunction is not 'new', but is rather a linear combination of those already presented. To see this, generalize the construction. The $\phi^{(p)}_{\lambda,k,r}(x)$ were built up explicitly from sine waves, but there is no particular reason to start with a sine-wave basis for the kernel. Consider, for example, the square wave or Haar wavelet

$$h(x) = \begin{cases} +1 & \text{for } 0 \le x < 1/2 \\ -1 & \text{for } 1/2 \le x < 1 \end{cases}$$

extended to the entire real-number line as a periodic function $h(x+n) = h(x)$ for integer n. The Haar wavelet is in the kernel of the Bernoulli operator: $\mathscr{L}_B h(x) = 0$, and thus can be used to create a fractal eigenfunction by means of the lattice shift equation 8.6:

$$\psi_k(x) = \sum_{n=0}^{\infty}\lambda^n h\left((2k+1)2^n x\right) \tag{8.9}$$

so that $\mathscr{L}_B \psi_k = \lambda \psi_k$ for any $|\lambda| < 1$ in the unit disk. Of course, c_λ is a special case: $c_\lambda(x) = \lambda(\lambda - 1)\psi_0(x)$, and so we see that c_λ is a linear combination of the functions 8.7. By the Fourier decomposition theorem, *any* periodic function odd about 1/2 will consist entirely of odd harmonics, and can thus be used to define an eigenfunction in the manner of eqn8.9. That it is a linear combination of the basis vectors 8.7 follows trivially.

There are a few practical reasons for choosing a basis other than the sine-wave basis. Most notably, convergence properties are altered: the Haar wavelet can avoid the Gibbs phenomenon of "ringing" that makes absolute bounds difficult to establish. The Haar wavelet also seems more compatible with the product topology of the lattice; it has a particularly simple form when considered as a function on the lattice.

Note, incidentally, that even more 'horrible' eigenfunctions can be constructed in this fashion. Consider, for example, $\delta\left(x - \frac{1}{4}\right) - \delta\left(x - \frac{3}{4}\right)$, extended to be periodic on the real line. It may be expressed as a Fourier series; it lies in the kernel of $\mathscr{L}_B$. It may be used to create a fractal eigenfunction: integrable but not square-integrable and certainly not bounded, thus showing that there are eigenfunctions of $\mathscr{L}_B$ that are in L_1 but not in L_2 or L_∞. This foreshadows some of the more challenging functions one encounters when analyzing other transfer operators, such as the GKW operator[28]. It also vindicates the focus on measurable functions on the product topology: the Dirac delta function is far easier to 'understand' as a measure on a topology than as a Fourier series.

8.5. **Fractal Symmetry and the Dyadic Monoid.** ————xxxxxxxxxxxx————xxxxxxxx

The fractal eigenfunctions are self-similar. This self-similarity follows directly from the shift invariance of the lattice construction, that is, of the translation invariance of the lattice Hamiltonian embodied in equation 8.3. Loosely speaking, one might say that translation-invariant Hamiltonians must necessarily have fractal eigenfunctions. The self-similarities have a non-trivial structure, which is exposed in this section.

For concreteness, consider first the action of the right-sift operators S_0 and S_1, defined by equation 8.4, acting on the $k = 0$ dyadic eigenfunction 8.9 built from the Haar wavelet. One has (dropping the subscript k fromψ):

$$\begin{aligned}
[S_0 \cdot \psi](x) &= \psi\left(\frac{x}{2}\right) \\
&= \sum_{n=0}^{\infty} \lambda^n h\left(2^{n-1}x\right) \\
&= h\left(\frac{x}{2}\right) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} h\left(2^{n-1}x\right) \\
&= h\left(\frac{x}{2}\right) + \lambda \psi(x) \\
&= -1 + \lambda \psi(x)
\end{aligned}$$

with the last step following because the Haar wavelet is identically -1 on the interval $0 \leq x < 1/2$. To emphasize this, write it more simply as

$$\psi\left(\frac{x}{2}\right) = -1 + \lambda \psi(x)$$

That is, one very explicitly has that the eigenfunction on the whole unit interval looks exactly like the eigenfunction on the half-interval, except that its scaled and offset: this is classic fractal self-similarity.

For the other shift operator S_1, one readily obtains that $S_1\psi = 1+\lambda\psi$. The action of these two shift operators has a representation in terms of 2x2 matrices acting on two-dimensional vectors. The basis vectors e_0 and e_1 for this vector space are

$$1\mapsto e_0=\left[\begin{array}{c}1\\0\end{array}\right]\quad\text{and}\quad\psi\mapsto e_1=\left[\begin{array}{c}0\\1\end{array}\right]$$

one has

$$S_0=\left[\begin{array}{cc}1&-1\\0&\lambda\end{array}\right]\quad\text{and}\quad S_1=\left[\begin{array}{cc}1&1\\0&\lambda\end{array}\right]\tag{8.10}$$

so that one has, for example $S_0e_1=-e_0+\lambda e_1$ as the action of the right shift on this vector space. The two matrices S_0 and S_1 generate a monoid, the *dyadic monoid*. A general element γ of the monoid is then

$$\gamma=S_0^{n_0}S_1^{n_1}S_0^{n_2}S_1^{n_3}\cdots$$

for non-negative integers $n_0,n_1,n_2,\cdots$. The monoid is free, in that there are no relations between S_0 and S_1, and each unique string of integers $n_0,n_1,n_2,\cdots$ results in a unique element of the monoid. For the purpose of this text, a *monoid* is defined much as a group, except that inverses are not considered. This is for two reasons: first, the inverses of S_0 and S_1 are not uniquely defined: the left and right inverse differ. Secondly, γ is to be understood to be the action of a monoid element acting on the unit interval. As long as the action maps the whole interval into a sub-interval, the action is well-defined; however, there are technical difficulties with going in the reverse direction.

The dyadic monoid has many interesting properties; these are discussed in greater detail in [30]. Among the notable properties is that the dyadic monoid is isomorphic to a certain subset of the Modular Group; this connection alone being important as the modular group plays an important role in number theory, specifically in the area of modular forms and automorphic forms. Not inconsequential is that the theory of modular forms again borders on the theory of the Riemann zeta. In a similar vein, one notes that the string of integers $[n_0,n_1,n_2,\cdots]$ is isomorphic to a continued fraction. Continued fractions are important both to the theory of Diophantine equations (Pellian equations), and in turn are seen to tie back into general phenomena seen in ergodic dynamics.

The integral of the Haar wavelet or square wave is the tent map or triangle wave

$$t(x)=\int_0^x h(y)\,dy=\begin{cases}\frac{1}{2}-2x & \text{when}\quad 0\le x\le 1/2\\ -\frac{3}{2}+2x & \text{when}\quad 1/2\le x\le 1\end{cases}$$

This is extended to the entire real line by defining $t(x+n)=t(x)$ for integer n. The triangle wave has the curious property of period-doubling upon iteration: that is, $t^k(x)=-t(2^{k-1}x)$ for $k>1$. The iterated tent map behaves as a shift state, in that the Bernoulli operator acts as an annihilation operator:

$$\left[\mathscr{L}_Bt^k\right](x)=t^{k-1}(x)$$

with the shift terminating at $k=1$:

$$\left[\mathscr{L}_Bt\right](x)=0$$

The fractal eigenfunction constructed from the triangle wave is the Takagi or blancmange curve:

$$b_\lambda(x)=\sum_{n=0}^{\infty}\lambda^n t(2^nx)$$

and so of course, $\mathscr{L}_B b_\lambda = \lambda b_\lambda$. There is a curious special case: the Takagi curve $b_{1/4}(x)$ is a parabola, corresponding to the Bernoulli polynomial $B_2(x)$. Apparently, the construction of the parabola by self-similar subdivision was known to Archimedes[22].

The blancmange curve transforms under a three-dimensional representation as the Haar-wavelet curve, in that

$$S_0 b_\lambda = -\left(x - \frac{1}{2}\right) + \lambda b_\lambda \quad \text{and} \quad S_1 b_\lambda = \left(x - \frac{1}{2}\right) + \lambda b_\lambda$$

Taking as the basis vectors

$$\begin{aligned} 1 \mapsto e_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ x \mapsto e_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ b_\lambda \mapsto e_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

the two generators may be written as

$$S_0 = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \lambda \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \lambda \end{bmatrix}$$

This process can be extended arbitrarily. One may choose to integrate the triangle wave, or to make a more general ansatz. Given some arbitrary polynomial $p(x)$, one may fashion a piece-wise polynomial periodic wave

$$q(x) = \begin{cases} -p(x) & \text{when} \quad 0 \le x < 1/2 \\ p\left(x - \frac{1}{2}\right) & \text{when} \quad 1/2 \le x < 1 \end{cases}$$

so that $q(x+1/2) = -q(x)$ or equivalently, $[\mathscr{L}_B q](x) = 0$. From this, one constructs the fractal wave

$$f_\lambda(x) = \sum_{n=0}^{\infty} \lambda^n q(2^n x)$$

which transforms as

$$S_0 f_\lambda = -p\left(\frac{x}{2}\right) + \lambda f_\lambda \quad \text{and} \quad S_1 f_\lambda = p\left(\frac{x}{2}\right) + \lambda f_\lambda$$

If the degree of the polynomial is m, then the associated linear representation is $m+2$ dimensional. One may choose as the basis vectors $e_k = x^k$ for $k \le m$ and $e_{m+1} = f_\lambda$. The matrix forms of S_0 and S_1 are obtained by expanding the polynomial $p(x/2)$ in terms of the chosen basis. There is nothing special about the monomial basis; one might consider any basis where e_k represents a polynomial of degree k.

As should be clear from the above construction, when $q(x)$ is not a polynomial, the representation cannot be finite-dimensional. In particular, the sine-wave based dyadic fractal 8.7 constructed from $q(x) = \exp 2\pi i x$ does not transform under a finite-dimensional representation.

The general dyadic case $k \neq 0$, given by

$$f_{\lambda,k}(x) = \sum_{n=0}^{\infty} \lambda^n q\left(2^n (2k+1) x\right)$$

does not change the situation. Since $[\mathscr{L}_B q](x) = 0$, one has $q(x+1/2) = -q(x)$, and so again, the action of S_0 and S_1 is given by

$$S_0 f_{\lambda,k} = q\left((2k+1)\frac{x}{2}\right) + \lambda f_{\lambda,k} \quad \text{and} \quad S_1 f_{\lambda,k} = -q\left((2k+1)\frac{x}{2}\right) + \lambda f_{\lambda,k}$$

As before, if $q(x)$ is made from a polynomial of degree m, the resulting fractal function transforms under an $m+2$ dimensional linear representation.

The general p-adic case proceeds in a similar manner. Polynomials of degree m transform as an $m+2$ dimensional representation, just as before. There are no longer two generators, but p generators $S_0, S_1, \cdots, S_{p-1}$.

XXXX Would deeper properties of the dyadic monoid be worth reviewing here???

XXXX – When are the inf-dimensional reps conjugate? What are conjugacy classes? etc.

XXXX more info on the p-adic free monoids, please.

8.6. **Number theoretic connections.** For prime number p, the number theoretic Hecke operator[2] may be written as

$$[T_p f](\tau) = p^{m-1} f(p\tau) + \frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{\tau+k}{p}\right)$$

In number theory, the Hecke operator is used in the classification of modular forms of order m. Now, in number theory, the above is intended to act only on functions meromorphic in the upper-half complex plane; *i.e.* for which τ lies in the upper half of the complex plane. However, we can immediately recognize the Hecke operator as the sum of the Bernoulli operator and its transpose:

$$T_p = p^{m-1}\mathscr{K}_p + \mathscr{L}_p$$

and so a certain form of the Hecke operator may be defined as acting on the unit interval and/or the Cantor set with the product topology. XXX expand, clarify.

Another curious operator is the Gauss-sum-like extension

$$\left[\mathscr{L}_p^{(m)} f\right](\sigma) = \frac{1}{p}\sum_{k=0}^{p-1} e^{2\pi i k m/p} f(S_k(\sigma))$$

regaining $\mathscr{L}_p = \mathscr{L}_p^{(0)}$.

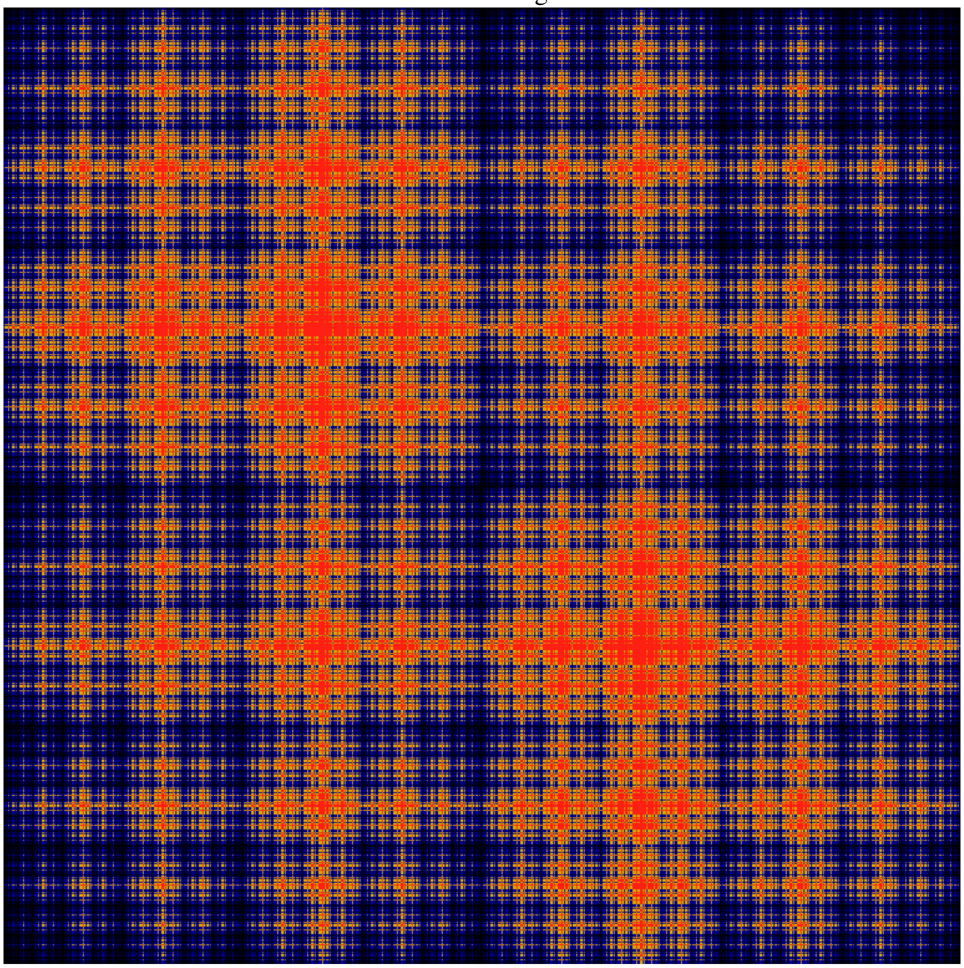
9. Two-Sided Lattice

The two-sided lattice is the lattice that stretches off to both the left and the right. The state of the two-sided lattice may be represented by two numbers $0 \le x, y \le 1$ with x given by equation **??**, and y likewise, but instead being a sum over the negative indexes, starting at $k = -1$. The energy is a simple sum of the energy of the half-lattices to the left and to the right, plus an interaction term involving spins on both the left and right. The probability density is a product of the probability densities on the left and right, adjusted by the interaction between the left and right sides.

The shift operator τ can be understood to be an operator that pops a bit off the right-hand half lattice, and pushes it onto the left half-lattice. In the dyadic notation, it can be recognized as the so-called *Baker's map*, a map from the unit square onto itself that cuts the square into two pieces, and stacks and squashes them. That is, given a pair of numbers (x, y) in the unit square, one has

$$\tau(x,y) = \left(\frac{x+\lfloor 2y\rfloor}{2}, 2y - \lfloor 2y\rfloor\right)$$

FIGURE 9.1. Ising Tartan



This tartan-like graph shows the Ising model probability density $P(\sigma)$ for the two-sided lattice using the dyadic mapping. That is, the lattice configuration $\sigma = (\sigma_{-N-1}, \cdots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_N)$ is represented by two numbers $0 \le x, y \le 1$ with

$$x(\sigma) = \sum_{k=0}^{N} \left(\frac{\sigma_k + 1}{2} \right) 2^{-(k+1)}$$

and

$$y(\sigma) = \sum_{k=0}^{N} \left(\frac{\sigma_{-k-1} + 1}{2} \right) 2^{-(k+1)}$$

The energy of a given configuration σ is computed using **??**, with the sum running from $-N$ to $+N$, of course. The probability density $P(\sigma)$ is given by equation **??**.

The graph here assumes the Ising potential, with $J = 0.3$ and $M = 0$ for a finite sized lattice with $N = 10$. The color choices here are such that black represents values where $P(\sigma) = P(x, y)$ are zero, blue are small values, with yellow and red being progressively larger values. This fractal tartan is invariant under the Baker's map.

The Baker's map can be studied in its own right as a discrete-time dynamical system, in that τ can be taken to define the time evolution of a unit square over a single time-step. The time evolution is invertible, in that the inverse map τ^{-1} is uniquely defined, being

$$\tau^{-1}(x,y) = \left(2x - \lfloor 2x \rfloor \, , \, \frac{y + \lfloor 2x \rfloor}{2}\right)$$

The inverse map can be seen to be identical to the forward map with x and y interchanged: that is, translation to the left on the lattice is identical to translation to the right, with left and right exchanged. One may say the map is PT-symmetric, with P standing for the "parity exchange" operation $x \leftrightarrow y$ and T being the "time inversion" operator $\tau \leftrightarrow \tau^{-1}$.

The two-sided lattice may be studied either as a lattice model, or as a dynamical system with time evolution given by τ. It is interesting to juxtapose these two viewpoints. As a lattice model, one is typically only interested in lattices for which the energy and the probability density are invariant under the action of the translation operator. That is, one is interested only in the classical Hamiltonians of eqn **??** which are translation-invariant:

$$H(\tau^n \sigma) = H(\sigma)$$

for all $n \in \mathbb{Z}$.

Considered as a dynamical system, one is typically interested in the time evolution of densities on the unit square, that is, of real-valued maps $\rho : [0,1] \times [0,1] \to \mathbb{R}$. The time evolution of a density can be simply understood as a physical model, where the density is a local density of some "dust" of points (x,y), with the time evolution of each point given by τ. Thus, the time evolution of this "dust" or density is given by the transfer operator $\mathscr{L}_T$ as

$$[\mathscr{L}_T \rho](x,y) = \left(\rho \circ \tau^{-1}\right)(x,y) = \rho\left(2x - \lfloor 2x \rfloor \, , \, \frac{y + \lfloor 2x \rfloor}{2}\right)$$

The transfer operator is thus a map $\mathscr{L}_T : \mathscr{F} \to \mathscr{F}$ where $\mathscr{F}$ is the set of functions on the unit square. Clearly $\mathscr{L}_T$ is a linear operator. Here, the subscript T on $\mathscr{L}_T$ is used to remind us that this is now the transfer operator on the two-sided lattice; it can be written as a combination of $\mathscr{L}_B$ and the Koopman operator $\mathscr{K}_B$ for the single-sided lattice. One is interested in characterizing the eigenfunctions and eigenvalues of $\mathscr{L}_T$.

The general idea behind equation **??** can be immediately appealed to, to construct some of the eigenstates of $\mathscr{L}_T$. That is, on considers functions $f(x,y)$ of the form

$$f = \sum_{n=-\infty}^{\infty} \lambda^n g \circ \tau^n \tag{9.1}$$

given some function $g(x,y)$ and number λ. Formally, such an f is an eigenstate of $\mathscr{L}_T$ with eigenvalue λ: that is, $\mathscr{L}_T f = \lambda f$. In practical terms, it is evident that not all possible λ can make the sum convergent, although one might expect that λ on the unit circle $|\lambda| = 1$ of the complex plane might lead to a convergent sum. Thus immediately, one deduces that $\mathscr{L}_T$ must surely be a unitary operator. The unitarity of $\mathscr{L}_T$ makes intuitive sense in a certain way: the map τ is invertible, and one expects time-reversible evolution to be described by unitary operators. The figure 9.1 shows a tartan-like distribution corresponding to $\lambda = 1$ and $g(x,y) = V_{\text{Ising}}(x)$ the Ising model potential. Some additional tartans, for other values of λ, are shown in figure xx.

The proper analysis of the transfer operator $\mathscr{L}_T$ requires that the function space $\mathscr{F}$ be pinned down more precisely. Classic results on time-symmetry breaking [16, 13, 6] indicate that whether or not $\mathscr{L}_T$ is unitary depends on the function space $\mathscr{F}$.

9.1. **Ladder operators.** Simple Harmonic Oscillator review.

There are several related, intertwined confusions, here. There lattice allows two different interpretations of what it means to be a ladder operator. XXX finish me.

10. MEASURE-THEORETIC DESCRIPTION

In this section, it will be shown that the transfer operator is the push-forward of the shift operator; a theorem and a sequence of lemmas will be posed, that hold in general form. The point of this theorem is to disentangle the role of topology, and specifically, the role of measure theory, from the use of the shift operator. We begin with a general setting.

Consider a topological space X, and a field F over the reals $\mathbb{R}$. Here, F may be taken to be $\mathbb{R}$ itself, or $\mathbb{C}$ or some more general field over $\mathbb{R}$. The restriction of F to being a field over the reals is is required, so that it can be used in conjunction with a measure; measures are always real-valued.

One may then define the algebra of functions $\mathscr{F}(X)$ on X as the set of functions $f \in \mathscr{F}(X)$ such that $f : X \to F$. An algebra is a vector space endowed with multiplication between vectors. The space $\mathscr{F}(X)$ is a vector space, in that given two functions $f_1, f_2 \in \mathscr{F}(X)$, their linear combination $af_1 + bf_2$ is also an element of $\mathscr{F}(X)$; thus f_1 and f_2 may be interpreted to be the vectors of a vector space. Multiplication is the pointwise multiplication of function values; that is, the product $f_1 f_2$ is defined as the function $(f_1 f_2)(x) = f_1(x) \cdot f_2(x)$, and so $f_1 f_2$ is again an element of $\mathscr{F}(X)$. Since one clearly has $f_1 f_2 = f_2 f_1$, multiplication is commutative, and so $\mathscr{F}(X)$ is also a commutative ring.

The space $\mathscr{F}(X)$ may be endowed with a topology. The coarsest topology on $\mathscr{F}(X)$ is the *weak topology*, which is obtained by taking $\mathscr{F}(X)$ to be the space that is topological dual to X. As a vector space, $\mathscr{F}(X)$ may be endowed with a norm $\|f\|$. For example, one may take the norm to be the L^p-norm

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$$

For $p = 2$, this norm converts the space $\mathscr{F}(X)$ into the Hilbert space of square-integrable functions on X. Other norms are possible, in which case $\mathscr{F}(X)$ has the structure of a Banach space rather than a Hilbert space.

Consider now a homomorphism of topological spaces $g : X \to Y$. This homomorphism induces the pullback $g^* : \mathscr{F}(Y) \to \mathscr{F}(X)$ on the algebra of functions, by mapping $f \mapsto g^*(f) = f \circ g$ so that $f \circ g : Y \to F$. The pullback is a linear operator, in that

$$g^*(af_1 + bf_2) = ag^*(f_1) + bg^*(f_2)$$

That the pullback is linear is easily demonstrated by considering how $g^* f$ acts at a point: $(g^* f)(x) = (f \circ g)(x) = f(g(x))$ and so the linearity of g^* on $af_1 + bf_2$ follows trivially.

One may construct an analogous mapping, but going in the opposite direction, called the push-forward: $g_* : \mathscr{F}(X) \to \mathscr{F}(Y)$. There are two ways of defining a push-forward. One way is to define it in terms of the sheaves of functions on subsets of X and Y. The sheaf-theoretic description is more or less insensitive to the ideas of measurability, whereas this is important to the definition of the transfer operator, as witnessed by the appearance of the Jacobian determinant in equation 2.2. By contrast, the measure-theoretic push-forward captures this desirable aspect. It may be defined as follows.

One endows the spaces X and Y with sigma-algebras $(X, \mathscr{A})$ and $(Y, \mathscr{B})$, so that $\mathscr{A}$ is the set of subsets of X obeying the axioms of a sigma-algebra, and similarly for $\mathscr{B}$. A mapping $g : X \to Y$ is called "measurable" if, for all Borel sets $B \in \mathscr{B}$, one has the pre-image $g^{-1}(B) \in \mathscr{A}$ being a Borel set as well. Thus, a measurable mapping induces a push-forward on the sigma-algebras: that is, one has a push-forward $g_*:\mathscr{F}(\mathscr{A})\to\mathscr{F}(\mathscr{B})$ given by $f\mapsto g_*(f)=f\circ g^{-1}$, which is defined by virtue of the measurability of g. The push-forward is a linear operator, in that

$$g_*(af_1+bf_2)=ag_*(f_1)+bg_*(f_2)$$

One regains the transfer operator as defined in equation 2.2 by considering the limiting behavior of the push-forward on progressively smaller sets. That is, one has

Theorem 2. *The transfer operator is the point-set topology limit of the measure-theoretic push-forward.*

Proof. The proof that follows is informal, so as to keep it simple. It is aimed mostly at articulating the language and terminology of measure theory. The result is none-the-less rigorous, if taken within the confines of the definitions presented.

Introduce a measure $\mu:\mathscr{A}\to\mathbb{R}^+$ and analogously $\nu:\mathscr{B}\to\mathbb{R}^+$. The mapping g is measure-preserving if ν is a push-forward of μ, that is, if $\nu=g_*\mu=\mu\circ g^{-1}$. The measure is used to rigorously define integration on X and Y. Elements of $\mathscr{F}(\mathscr{A})$ can be informally understood to be integrals, in that $f(A)$ for $A\in\mathscr{A}$ may be understood as

$$f(A)=\int_A\tilde{f}(z)d\mu(z)=\int_A\tilde{f}(z)\left|\mu'(z)\right|dz$$

where $|\mu'(x)|$ is to be understood as the Jacobean determinant at a point $x\in X$. Here, $\tilde{f}$ can be understood to be a function that is being integrated over the set A, whose integral is denoted by $f(A)$. The value of $\tilde{f}$ at a point $x\in X$ can be obtained by means of a limit. One considers a sequence of $A\in\mathscr{A}$, each successively smaller than the last, each containing the point x. One then has

$$\lim_{\overrightarrow{A\ni x}}\frac{f(A)}{\mu(A)}=\tilde{f}(x)$$

which can be intuitively proved by considering A so small that $\tilde{f}$ is approximately constant over A:

$$f(A)=\int_A\tilde{f}(z)d\mu(z)\approx\tilde{f}(x)\int_A d\mu=\tilde{f}(x)\mu(A)$$

To perform the analogous limit for the push-forward, one must consider a point $y\in Y$ and sets $B\in\mathscr{B}$ containing y. In what follows, it is now assumed that $g:X\to Y$ is a multi-sheeted countable covering of Y by X. By this it is meant that for any y that is not a branch-point, there is a nice neighborhood of y such that its pre-image consists of the union of an at most countable number of pair-wise disjoint sets. That is, for y not a branch point, and for $B\ni y$ sufficiently small, one may write

$$g^{-1}(B)=A_1\cup A_2\cup\cdots=\bigcup_{j=1}^{k}A_j$$

where k is either finite or stands for ∞, and where $A_i\cap A_j=\varnothing$ for all $i\neq j$. At branch points, such a decomposition may not be possible. The axiom of sigma-additivity guarantees that such multi-sheeted covers behave just the way one expects integrals to behave: in other words, one has

$$\mu\left(g^{-1}(B)\right)=\mu\left(\bigcup_{j=1}^{k}A_j\right)=\sum_{j=1}^{k}\mu\left(A_j\right)$$

whenever the collection of A_j are pair-wise disjoint. Similarly, in order to have the elements $f \in \mathscr{F}(\mathscr{A})$ behave as one expects integrals to behave, one must restrict $\mathscr{F}(\mathscr{A})$ to contain only sigma-additive functions as well, so that

$$f\left(g^{-1}(B)\right) = f\left(\bigcup_{j=1}^{k} A_j\right) = \sum_{j=1}^{k} f(A_j)$$

As the set B is taken to be smaller and smaller, the sets A_j will become smaller as well. Denote by x_j the corresponding limit point of each A_j, so that $g(x_j) = y$ and the pre-image of y consists of these points: $g^{-1}(y) = \left\{x_1, x_2, \cdots \,\middle|\, g(x_j) = y\right\}$. One now combines these provisions to write

$$\begin{aligned} \left[g_*\tilde{f}\right](y) &= \lim_{\overrightarrow{B \ni y}} \left[\frac{(g_* f)(B)}{\nu(B)}\right] \\ &= \lim_{\overrightarrow{B \ni y}} \left[\frac{\left(f \circ g^{-1}\right)(B)}{\nu(B)}\right] \\ &= \lim_{\overrightarrow{A_j \ni g^{-1}(y)}} \frac{f(A_1 \cup A_2 \cup \cdots)}{\nu(B)} \\ &= \lim_{\overrightarrow{A_j \ni g^{-1}(y)}} \frac{\sum_{j=1}^{k} f(A_j)}{\nu(B)} \\ &= \sum_{j=1}^{k} \tilde{f}(x_j) \lim_{\overrightarrow{A_j \ni x_j}} \frac{\mu(A_j)}{\nu(B)} \qquad (10.1) \end{aligned}$$

The limit in the last line of this sequence of manipulations may be interpreted in two ways, depending on whether one wants to define the measure ν on Y to be the push-forward of μ, or not. If one does take it to be the push-forward, so that $\nu = g_*\mu$, then one has

$$\lim_{\overrightarrow{A_j \ni x_j}} \frac{\mu(A_j)}{g_*\mu(B)} = \frac{1}{\left|g'(x_j)\right|}$$

where $\left|g'(x_j)\right|$ is the Jacobian determinant of g at x_j. This last is a standard result of measure theory, and can be intuitively proved by noting that $g(A_j) = B$, so that

$$\nu(B) = \int_{A_j} g'(z)\, d\mu(z) \approx g'(x_j)\, \mu(A_j)$$

for "small enough" B. Assembling this with the previous result, one has

$$\left[g_*\tilde{f}\right](y) = \sum_{x_j \in g^{-1}(y)} \frac{\tilde{f}(x_j)}{\left|g'(x_j)\right|} \qquad (10.2)$$

which may be easily recognized as equation 2.2. This concludes the proof of the theorem, that the transfer operator is just the point-set topology limit of the push-forward. □

In simplistic terms, the push-forward can be thought of as a kind of change-of-variable. Thus, one should not be surprised by the following lemma, which should be recognizable as the Jacobian, from basic calculus.

Lemma 3. *(Jacobian) One has*

$$\sum_{x_j \in g^{-1}(y)} \frac{1}{\left|g'(x_j)\right|} = 1$$

Proof. This follows by taking the limit $\overrightarrow{A_j \ni x_j}$ of

$$\frac{\mu(A_j)}{g_*\mu(B)} = \frac{\mu(A_j)}{\sum_{i=1}^{k}\mu(A_i)}$$

and then summing over j. □

Corollary 4. *The uniform distribution (*i.e. *the measure) is an eigenvector of the transfer operator, associated with the eigenvalue one.*

Proof. This may be proved in two ways. From the viewpoint of point-sets, one simply takes $\tilde{f} =$ const. in equation 10.2, and applies the lemma above. From the viewpoint of the sigma-algebra, this is nothing more than a rephrasing of the starting point, that $\nu = g_*\mu$, and then taking the space $Y = X$, so that the push-forward induced by $g : X \to X$ is a measure-preserving map: $g_*\mu = \mu$. □

The last corollary is more enlightening when it is turned on its side; it implies two well-known theorems, which follow easily in this framework.

Corollary 5. *(Ruelle-Perron-Frobenius theorem). All transfer operators are continuous, compact, bounded operators; furthermore, they are isometries of Banach spaces.*

Proof. This theorem is of course just the Frobenius-Perron theorem, recast in the context of measure theory. By definition, the measures have unit norm: that is, $\|\mu\|_1 = 1$ and $\|\nu\|_1 = 1$. This is nothing more than the statement that the spaces X and Y are measurable: the total volume of X and Y is, by definition, one. Since $\nu = g_*\mu$, we have $\|g_*\mu\|_1 = 1$, and this holds for all possible measures $\mu \in \mathscr{F}(X)$.

Recall the definition of a bounded operator. Given a linear map $T : U \to V$ between Banach spaces $U$and V, then T is bounded if there exists a constant $C < \infty$ such that $\|Tu\|_V \leq C\|u\|_U$for all $u \in U$. But this is exactly the case above, with $T = g_*$, and $U = \mathscr{F}(X)$, $V = \mathscr{F}(Y)$, and $C = 1$. The norm of a bounded operator is conventionally defined as

$$\|T\| = \sup_{u\neq 0}\frac{\|Tu\|_V}{\|u\|_U} = \sup_{\|u\|_U\leq 1}\|Tu\|_V$$

and so we have the norm of g_* being $\|g_*\| = 1$. That g_* is an isometry follows trivially from $\|g_*\mu\|_1 = \|\nu\|_1$ and that g_* is linear. □

The corollary 4 can also be treated as a corollary to the Perron-Frobenius theorem: namely, that there is at least one vector that corresponds to the maximum eigenvalue of g_*. This eigenvector is in fact the Haar measure, as the next theorem shows.

Theorem 6. *(Haar measure) For any homomorphism $g : X \to X$, one may find a measure μ such that $g_*\mu = \mu$; that is, every homomorphism g of X induces a measure μ on X such that g is a measure-preserving map. If g is ergodic, then the measure is unique.*

Proof. By definition, μ is a fixed point of g_*. The fixed point exists because $g_*$is a bounded operator, and the space of measures is compact, and so a bounded operator on a compact space will have a fixed point. The existence of the fixed point is given by the Markov-Kakutani theorem[7, p 456]. The Markov-Kakutani theorem also provides the uniqueness condition: if there are other push-forwards h_* that commute with g_*, then each such push-forward will also have a fixed point. The goal is then to show that when g is ergodic, there are no other functions h that commute with g. But this follows from the definition of ergodicity: when g is ergodic, there are no invariant subspaces, and the orbit of g is the whole space. As there are no invariant subspaces, there are no operators that can map between these subspaces, *i.e.* there are no other commuting operators. □

A peculiar special case is worth mentioning: if g is not ergodic on the whole space, then typically one has that the orbit of g splits or foliates the measure space into a bunch of pairwise disjoint leaves, with g being ergodic on each leaf. The Markov-Kakutani theorem then implies that there is a distinct fixed point μ in each leaf, and that there is a mapping that takes μ in one leaf to that in another.

In the language of dynamical systems, the push-forward g_* is commonly written as $\mathscr{L}_g$, so that one has

$$g_* = \mathscr{L}_g : \mathscr{F}(X) \to \mathscr{F}(X)$$

now being called the transfer operator or the Ruelle-Frobenius-Perron operator.

In the language of physics, the fixed point μ is called the "ground state" of a system. When it is unique, then the ground state is not degenerate; when it is not unique, then the ground state is said to be degenerate. The operator g_* is the time-evolution operator of the system; it shows how physical fields $f \in \mathscr{F}(X)$ over a space X evolve over time. When F is the complex numbers $\mathbb{C}$, the fact that $\|g_*\| = 1$ is essentially a way of stating that time-evolution is unitary; the Frobenius-Perron operator is the unitary time-evolution operator of the system. What is called "second quantization" in physics should be interpreted as the fitting of the space $\mathscr{F}(X)$ with a set of basis vectors, together with a formulation of g_* in terms of that basis.

XXX ToDo: This means the Koopman operator is the pullback. Since pullback and push-forward are category-theoretic adjoints, nail down more of the category theory bits & pieces. Make it tight. Probably split this off into its own paper ...

11. The Topological Zeta

XXX ToDo: explain topo zeta gets this has this name, and why this concept is important. XXX

The topological zeta of the Bernoulli operator can be computed very easily in the polynomial basis because we know the eigenvalues and these form a simple series. We'll define the Bernoulli topological zeta as

$$\zeta_B(t) \equiv \frac{1}{\det\left[\mathbb{I} - t\mathscr{L}_B\right]}$$

We start by noting its inverse:

$$\begin{aligned}
\det\left[\mathbb{I} - t\mathscr{L}_B\right] &= \prod_{n=0}^{\infty}\left(1 - t\,2^{-n}\right) \\
&= 1 - t\sum_{j=0}^{\infty} 2^{-j} + t^2 \sum_{j=0}^{\infty} 2^{-j} \sum_{\substack{k=0 \\ k\neq j}}^{\infty} 2^{-k} - t^3 \ldots \\
&= 1 - 2t + \frac{8}{3}t^2 - \frac{16}{7}t^3 + \frac{128}{105}t^4 - \ldots \qquad (11.1)
\end{aligned}$$

Successive terms of this series are hard to compute, and it would be interesting to know what the generating function for this series is. The series appears to have a circle of convergence of radius one. The zeta can be computed directly by working with its logarithm:

$$\begin{aligned}
\log \zeta_B(t) &= \log \prod_{n=0}^{\infty} \left(1 - t\, 2^{-n}\right)^{-1} \\
&= -\sum_{n=0}^{\infty} \log\left(1 - t\, 2^{-n}\right) \\
&= \sum_{k=1}^{\infty} \frac{t^k}{k} \frac{2^k}{2^k - 1} \\
&= -\log(1-t) + \sum_{k=1}^{\infty} \frac{t^k}{k} \frac{1}{2^k - 1} \qquad (11.2)
\end{aligned}$$

Thus we have $\mathrm{Tr}\mathscr{L}_B^k = 2^k/(2^k-1)$. Of some curiosity is the proximity of the Erdos-Borwein constant:

$$\begin{aligned}
1.6066... &= \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \\
&= \sum_{n=1}^{\infty} \frac{d(n)}{2^n} \qquad (11.3)
\end{aligned}$$

which marks the first appearance of a classical number-theoretic function in the proceedings so far: $d(n)$ is the number of divisors of n. This arises from the Lambert series

$$\sum_{n=1}^{\infty} d(n) x^n = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}$$

The sum

$$E(t) = \sum_{k=1}^{\infty} \frac{t^k}{1 - 2^{-k}}$$

can be re-summed as a Lambert series, namely,

$$E(t) = \sum_{k=1}^{\infty} b_k 2^{-k}$$

where

$$b_k = \sum_{n|k} (2t)^n$$

The analytic/meromorphic structure of this zeta is not clear; its dull within the unit disk, and its not quite obvious what the continuation is outside of the disk. XXX ToDo: get the full analytic structure.

12. Curiosities

We list here some intriguing forms that suggest further relationships.

The Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(n)$ obeys a *dimidiation formula* that is reminiscent of the Bernoulli map:

$$\begin{aligned}(a)_{2n} &= 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n \\ (a)_{2n+1} &= 2^{2n+1}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n\end{aligned}$$

13. CONCLUSIONS

Apologies for the format of this paper.

APPENDIX A. REVIEW OF DIRAC NOTATION

This appendix reviews the Dirac or bra-ket notation on the polynomial and square-integrable spaces, and illustrates this by performing a calculation that mixes the two.

In section 4, we constructed the polynomial space, and its dual. In Dirac notation, the basis vectors are the monomials:

$$\langle x|n\rangle = x^n$$

while the dual vectors are the linear functionals

$$\langle n|x\rangle = \frac{(-1)^n}{n!}\delta^{(n)}(x)$$

That these are truly dual is expressed by equations 4.2 and 4.3; namely, that $\langle n|m\rangle = \delta_{mn}$ and that $\mathbb{I}_A = \sum_{n=0}^{\infty} |n\rangle\langle n|$. The later may be understood as the identity operator on the space of polynomials, or, more broadly, on the space of analytic functions. This interpretation is justified by realizing that it is just a formalized notation for the Taylor's series in eqn 4.5. The monomials, and their duals, truly provide a basis for the space of analytic functions.

Let us now review the analogous situation, this time for the Hilbert space L_2 of square-integrable functions. The review is remedial but is needed to present the surprising results. One writes, for some (periodic) function $f(x)$, the Fourier Series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx \ + b_n \sin 2\pi nx$$

where the conjugates of f are given by

$$a_n = 2\int_0^1 f(x)\,\cos(2\pi nx)\,dx$$

and

$$b_n = 2\int_0^1 f(x)\,\sin(2\pi nx)\,dx$$

The sine and cosine notation is somewhat awkward, and it will be easier to use the complex exponential $\exp(i2\pi mx) = \cos(i2\pi mx) + i\sin(i2\pi mx)$. At the same time, let us move over to bra-ket notation. The $|em\rangle$ will denote the basis vectors for L_2. These may be defined in terms of their components in coordinate space, namely $\langle x|em\rangle = \exp(i2\pi mx)$. The conjugate vectors $\langle en|$ have an equally simple representation: $\langle en|x\rangle = \exp(-i2\pi nx)$. These are orthogonal over coordinate space in the usual sense:

$$\langle em|en\rangle = \int_0^1 dx\,\langle em|x\rangle\langle x|en\rangle = \int_0^1 dx\,\exp(2\pi i(n-m)x) = \delta_{nm}$$

These are also complete over coordinate space, in that, for any arbitrary square-integrable coordinate-space function $f(x) = \langle x|f\rangle$, one has

$$\begin{aligned}
f(x) &= \int_0^1 dy\,\delta(x-y)\,f(y) \\
&= \sum_{n=-\infty}^{\infty} \exp(i2\pi nx) \int_0^1 dy\,\exp(-i2\pi ny) f(y) \\
&= \sum_{n=-\infty}^{\infty} \exp(i2\pi nx) \int_0^1 dy\,\langle en|y\rangle \langle y|f\rangle \\
&= \sum_{n=-\infty}^{\infty} \langle x|en\rangle \langle en|f\rangle \\
&= \langle x| \sum_{n=-\infty}^{\infty} |en\rangle \langle en|\,|f\rangle \\
&= \langle x|\,1_F\,|f\rangle \\
&= \langle x|f\rangle
\end{aligned}$$

The above made use of the usual expansion of the Dirac delta as a Fourier series:

$$\delta(x-y) = \sum_{n=-\infty}^{\infty} \exp\left(i2\pi n(x-y)\right)$$

The completeness of the basis over L_2 may be expressed as a decomposition of the identity operator on L_2, namely $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$. Here, the subscript F reminds us of the Fourier-series nature of this identity operator. The identity operator has the matrix elements that one expects in both the Fourier space and in coordinate space: that is, $\langle em|\,1_F\,|en\rangle = \delta_{nm}$ and $\langle x|\,1_F\,|y\rangle = \delta(x-y)$.

So far, all of the above definitions and developments are standard textbook fare. We now embark on a slightly more challenging calculation: to show that $\langle m|\,1_F\,|n\rangle = \delta_{nm}$. In the process, one encounters some divergent sums; these can, however, be rescued by means of analytic continuation.

Begin by writing the components of the vector $|em\rangle$ in the polynomial-space representation. For $m \neq 0$, one has

$$\begin{aligned}
\langle n|em\rangle &= \int_0^1 dx\,\langle n|x\rangle \langle x|em\rangle \\
&= \int_0^1 dx\,\frac{(-1)^n}{n!}\,\delta^{(n)}(x)\,e^{i2\pi mx} \\
&= \int_0^1 dx\,\frac{\delta(x)}{n!}\,\frac{d^n}{dx^n}\,e^{i2\pi mx} \\
&= \frac{(i2\pi m)^n}{n!}
\end{aligned}$$

while $\langle n|e0\rangle = \delta_{n0}$. Essentially, this is nothing more than a plain-old Taylor's series expansion of the exponential function; reasonable, insofar as the exponential function is an entire function on the complex plane. The first step in the above was to insert the coordinate-space identity operator:

$$1_C = \int_0^1 dx\,|x\rangle \langle x|$$

That 1_C deserves the status of an identity operator may be seen by noting that, for any function f, one has $f(y) = \langle y|f\rangle = \int_0^1 dx \langle y|x\rangle \langle x|f\rangle = \int_0^1 dx \delta(y-x) f(y)$, where we made use of $\langle y|x\rangle = \delta(y-x)$.

The conjugate vectors have a slightly trickier form. They are the Fourier components of monomials. For $m \neq 0$, $n \neq 0$ one has

$$\begin{aligned}\langle em|n\rangle &= \int_0^1 dy \langle em|y\rangle \langle y|n\rangle \\ &= \int_0^1 \exp(-2\pi imy)\, y^n\, dy \\ &= \frac{-1}{2\pi im} + \frac{n}{2\pi im}\int_0^1 \exp(-2\pi imy)\, y^{n-1}\, dy \\ &= -\frac{1}{2\pi im}\sum_{k=0}^{n-1} \frac{n!}{(n-k)!}\left(\frac{1}{2\pi im}\right)^k\end{aligned}$$

while, for $m = 0$, one simply has $\langle e0|n\rangle = 1/(n+1)$, and, for $n = 0$, one has $\langle em|0\rangle = \delta_{m0}$. Let us now try to explicitly evaluate the matrix elements of the Fourier identity operator 1_F in the polynomial representation. That is, we attempt to write the matrix elements of $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$. So, by direct substitution,

$$\begin{aligned}\langle p|\, 1_F\, |n\rangle &= \sum_{m=-\infty}^{\infty} \langle p|em\rangle \langle em|n\rangle \\ &= \sum_{m=-\infty}^{\infty} \left[\delta_{p0} + (1-\delta_{p0})\frac{(2\pi im)^p}{p!}\right]\left[\frac{\delta_{m0}}{n+1} - \frac{(1-\delta_{m0})}{2\pi im}\sum_{k=0}^{n-1}\frac{n!}{(n-k)!}\left(\frac{1}{2\pi im}\right)^k\right]\end{aligned}$$

Consider first the matrix element $n = 1$, $p \neq 0$, to see the misery of this expression:

$$\langle p \neq 0|\, 1_F\, |n = 1\rangle = -\frac{(2\pi i)^{p-1}}{p!}\sum_{m=1}^{\infty}\left[m^{p-1} - (-m)^{p-1}\right]$$

For p even, the sum on the right vanishes. For p even, the sum on the right is formally divergent. One may rescue the situation by making the Ansatz that the summation should have been replaced by $\zeta(1-p)$, *i.e.* that the summation should be replaced by it's analytic continuation, which is the Riemann zeta function. So:

$$\langle p \neq 0|\, 1_F\, |n = 1\rangle = -\frac{2\,(2\pi i)^{p-1}}{p!}\zeta(1-p)$$

For $p = 1$,we have $\zeta(0) = -1/2$ and so $\langle p = 1|\, 1_F\, |n = 1\rangle = 1$ as expected. For p odd, one has that $\zeta(-2k) = 0$ for all positive integers $k > 0$, and so one has exactly what was expected: $\langle p \neq 0|\, 1_F\, |n = 1\rangle = \delta_{p1}$. Calculations for other values of n are somewhat more difficult, but ultimately provide the expected answer: $\langle p|\, 1_F\, |n\rangle = \langle p|n\rangle = \delta_{pn}$. There is, however, a cautionary tale that becomes clear as one tries to carry out this calculation: the manipulations can become painful, and formally divergent sums are best avoided beforehand, by re-arranging terms before they become a problem.

APPENDIX B. THE KOOPMAN OPERATOR

The Koopman operator is in a certain sense conjugate to the Frobenius-Perron operator, and defines how observables evolve. Given a density $\rho(x)$ we say that the observation of a function $f(x)$ by ρ is

$$\langle f \rangle_\rho = \int_0^1 f(x)\rho(x)\,dx$$

The term "observable" comes from usage in quantum mechanics, where $f(x)$ is associated with the eigenvalues of an operator. We do not need to appeal to these operator equations for the following development. The Koopman operator K gives the change in f when U acts on ρ, thus:

$$K_g : \langle f \rangle_\rho \to \left\langle K_g f \right\rangle_\rho = \int_0^1 [K_g f](x)\rho(x)\,dx = \int_0^1 f(x)[U_g\rho](x)\,dx$$

In Dirac bra-ket notation, we have

$$\begin{aligned}\int_0^1 f(x)[U_g\rho](x)\,dx &= \int_0^1 \left\langle x|U_g|\rho\right\rangle \langle x|f\rangle\; dx \\ &= \int_0^1 dx \int_0^1 dy \left\langle x|U_g|y\right\rangle \langle y|\rho\rangle \langle x|f\rangle\; dx\end{aligned}$$

and so we have

$$[K_g f]\,(y) = \int_0^1 \left\langle x|U_g|y\right\rangle \langle x|f\rangle\; dx = \int_0^1 U_g(x,y) f(x)\,dx = \int_0^1 \delta\,(x - g(y))\, f(x)\,dx$$

This gives the action of the Koopman operator in a coordinate-space representation. As is the recurring theme, different representations can lead to different results. In the coordinate-space representation, the Koopman operator appears to be the transpose of the Frobenius-Perron operator, in that $K(x,y) = U(y,x)$. However, in a general representation, whether the Koopman operator is the transpose or the complex conjugate or something else needs to be determined on a case-by-case basis, with an appeal to the particular operator $g(x)$ and the representations on which it works.

Appendix C. Topologically Conjugate Maps

Conjugation of the function that generates the map will provide, in general, another map that behaves exactly the same as the first, as long as the conjugating function is a 1-1 and onto diffeomorphism. That is, if ϕ is invertible, so that

$$\gamma = \phi \circ g \circ \phi^{-1}$$

then γ will iterate the same way that g does: $\gamma^n = \phi \circ g^n \circ \phi^{-1}$. The orbit of any point x under the map g is completely isomorphic to the orbit of a point $y = \phi(x)$ under the map γ. Because the (chaotic) point dynamics of these two maps are isomorphic, we expect just about any related construction and analysis to show evidence of this isomorphism.

In particular, we expect that the Koopman and Frobenius-Perron operators for γ are conjugate to those for g:

$$U_\gamma = U_\phi^{-1} U_g U_\phi$$

XXX ToDo derive the above. Show that eigenvalues are preserved. XXX A good derivation is given in the GKW paper. The most trivial way to see that the eigenvalues are unchanged is through the formal definition of the characteristic polynomial for this operator, which is

$$p_U(\lambda) = \det\left[U_g - \lambda \mathbb{I}\right]$$

Just as in the finite-dimensional case, a similarity transform commutes inside the determinant, leaving the characteristic polynomial unchanged. XXX ToDo a more correct, non-formal proof that the eigenvalues are preserved. Note, as the GKW paper notes, that although one can still have similarity xforms when fun is not diffeomorphic, that, in such a case, the eigenvalues are not preserved.

Note that in the construction of this proof, we invoke the Jacobian $|d\phi(y)/dy|_{y=\phi^{-1}(x)}$ and thus, in order to preserve the polynomial-rep eigenvalues, the conjugating function must be a diffeomorphism; a homeomorphism does not suffice. One can have conjugate maps with completely isomorphic point dynamics, but the eigenvalue spectra associated with these maps will *not* be identical. Am explicit example is the GKW operator, which is isomorphic to the dyadic sawtooth operator, but is not diffeomorphic. The two operators have distinct spectra.

APPENDIX D. THE TOPOLOGICAL ZETA

Another interesting quantity is the topological zeta function associated with the transfer operator. It is formally defined by

$$\zeta_{U_g}(t)=\frac{1}{\det\left[\mathbb{I}-tU_g\right]}$$

and embeds number-theoretic information about the map. Using standard formal manipulations on operators, one can re-write the above as the operator equation

$$\zeta_{U_g}(t)=\exp\sum_{k=1}^{\infty}\frac{t^k}{k}\,\mathrm{Tr}U_g^k$$

Of associated interest is the Maclaurin Series

$$t\frac{d}{dt}\log\zeta_{U_g}(t)=\sum_{k=1}^{\infty}n_kt^k$$

where we can read off $n_k=\mathrm{Tr}U_g^k$. From graph theory and the theory of dynamical systems, it is known that the n_k correspond to the number of periodic orbits of length k. In the context of dynamical systems, this zeta is often referred to as the Artin-Mazur Zeta function. In the context of graph theory, it is referred to as the Ihara Zeta. Both are connected to the Selberg Zeta.

The standard definition of the Ihara Zeta applies only to the adjacency matrix of finite-sized graphs. Adjacency matrices only have (non-negative) integer entries as matrix elements. Thus, we ask: given an appropriate basis, can an infinite-dimensional transfer operator be written so as to have integer entries as matrix elements?

The standard definition of the Artin-Mazur Zeta function requires that the number of fixed points (periodic orbits) be a finite number. For the operators that we are studying, there will in general be (countably) infinite number of periodic orbits. Yet the zeta will still be well defined, although the coefficients of the Maclaurin expansion will not be integers. Can these be reinterpreted as a density or measure?

APPENDIX E. JONQUIÈRE'S IDENTITY

This appendix establishes the relationship between the functions $\beta(x,s)$ and the Hurwitz zeta function

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}$$

This relationship is well-known; for example, it is reviewed in the Bateman manuscripts in the guise of an identity on the Lerch transcendental[9]; see also Wikipedia on Polylogarithms[34]. As a relation on polylogarithms, it is known as Jonquière's identity[20, Section 7.12.2][17].

One may confirm this by following a very old-fashioned recipe for obtaining the functional relation for a zeta-like sum. Start by expressing the gamma function as

$$\int_0^\infty dy\, e^{-2\pi ny} y^{s-1} = \frac{\Gamma(s)}{(2\pi n)^s}$$

Substituting into the expression for β and performing the sum, one may write

$$\beta(x;s) = 2s \int_0^\infty dy\, \frac{y^{s-1}}{\exp\left(-2\pi i(x+iy)\right) - 1}$$

Then, following a traditional trick [8, pp 13 ff], re-write this as a contour integral

$$\beta(x;s) = \frac{-is}{\sin \pi s} \oint \frac{(-y)^s}{\exp\left(-2\pi i(x+iy)\right) - 1}\, \frac{dy}{y}$$

where the contour is taken to extend from $+\infty + i\varepsilon$, running just above the positive real axis, to the origin, circling the origin in a clockwise fashion, and returning to $+\infty - i\varepsilon$ just under the real axis. The contour essentially encloses the cut of the logarithm in the expression $(-y)^s = \exp s \log(-y)$. The old fashioned recipe calls for closing the contour at infinity (in a counter-clockwise direction) and then taking the dubious step of asserting Cauchy's Theorem to equate the integral around the cut to the sum of the poles, where we note that we have a pole whenever $x + iy = n$ for some integer n. By doing this we get the formal summation

$$\frac{i \sin \pi s}{s} \beta(x;s) = \exp\left(\frac{i\pi s}{2}\right) \sum_{n=-\infty}^{\infty} (n-x)^{s-1}$$

This is a "formal sum", since the preceding steps required taking $\Re s > 1$ whereas now one needs to take $\Re s < 0$. This is a bit of jiggery-pokery that is common for this type of presentation; a different set of tools is required to do better. So we proceed, ignoring these difficulties. Re-write this sum as

$$\frac{i \sin \pi s}{s} \beta(x;s) = \exp\left(\frac{i\pi s}{2}\right) \left[\sum_{n=0}^{\infty} (n + (1-x))^{s-1} + e^{-i\pi(s-1)} \sum_{n=0}^{\infty} (n+x)^{s-1} \right]$$

where we were mindful to rotate counter-clockwise for $n < 0$ when replacing $(-)^n$ by $e^{-i\pi n}$ instead of the sloppy and incorrect $e^{i\pi n}$. Recognizing the sums as the Hurwitz Zeta, this then gives the desired result:

$$\beta(x;s) = \frac{is}{\sin \pi s} \left[e^{-i\pi s/2} \zeta(1-s,x) - e^{i\pi s/2} \zeta(1-s,1-x) \right]$$

It is straightforward to invert this and solve for ζ; one gets

$$\zeta(1-s,x) = \frac{1}{2s} \left[e^{-i\pi s/2} \beta(x;s) + e^{i\pi s/2} \beta(1-x;s) \right]$$

thus proving the assertion that the Hurwitz zeta is an eigenfunction of the Bernoulli operator, with eigenvalue 2^{s-1}. To verify the correctness of the above steps, expand the exponentials in terms of their real and imaginary parts, to find that

$$\zeta(z,x) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{1-z}} \right]$$

which agrees with standard textbook presentations of the Hurwitz zeta; see for example [1, Thm 12.6, Ex 12.2].

APPENDIX F. VISUALIZING THE ZETA FUNCTION

This section provides a visualization and a simple discussion of the analytic properties of the zeta eigenfunctions. The main point made is that the eigenfunctions are smooth, that is, infinitely differentiable (C^∞) in x for all x except at the endpoints $x = 0,1$, where there is an essential singularity. There are eigenfunctions that have eigenvalues greater than one; these, while quite smooth, are not square-integrable: they are divergent at $x = 0,1$. However, in all other respects, the eigenfunctions are analytically well-behaved, even if a bit "lumpy" and uneven, as the following graphs show.

There are eigenfunctions with eigenvalues greater than one, essentially because the Hurwitz zeta can be analytically continued to everywhere on the complex plane except for a simple pole at $s = 1$. Examining these eigenfunctions, one quickly discovers that these are not square-integrable: they have singularities located at $x = 0,1$. That is, for for $\Re s > 0$, the Hurwitz zeta $\zeta(s,x)$ has a clear singularity x^{-s} at $x = 0$. Separating this out explicitly, one has

$$\frac{\sin\pi s}{is}\beta(x;s) = \frac{e^{-i\pi s/2}}{x^{1-s}} - \frac{e^{i\pi s/2}}{(1-x)^{1-s}} + e^{-i\pi s/2}\left(\zeta(1-s,1+x)\right) - e^{i\pi s/2}\left(\zeta(1-s,2-x)\right)$$

The first part of the equation above encapsulates the singularities at $x = 0,1$ that occur when working with eigenvalues $|\lambda| = |2^{-s}| > 1$, that is, with $\Re s < 0$. The remaining term is well-behaved and is shown in figure F.3.

When $|\lambda| = |2^{-s}| < 1/2$, that is, when $\Re s > 1$, there is no singularity, and $\beta(x;s)$ is finite on the entire interval $x \in [0,1]$, including the endpoints. For $1/2 < \Re s \leq 1$ there is a bit of funny-business at the endpoints; that is, there is a weak divergence there, but the function overall remains square-integrable. Things break loose after that, with the exception of $s = 0$, where we have $\beta(x;0) = -1$, a constant independent of x. This essentially follows from the nature of differentiation on the Bernoulli polynomials, which we'll see below. For s near zero, the function $\beta(x;s)$ has severe ringing artifacts in x, suffering from a variation of the Gibbs phenomenon.

The function $\beta(x;s)$ is C^∞ for $x \in (0,1)$ but not at the endpoints $x = 0,1$. This can be easily seen by writing the derivative

$$\frac{d}{dx}\beta(x;s) = 2\pi i\beta(x;s-1)$$

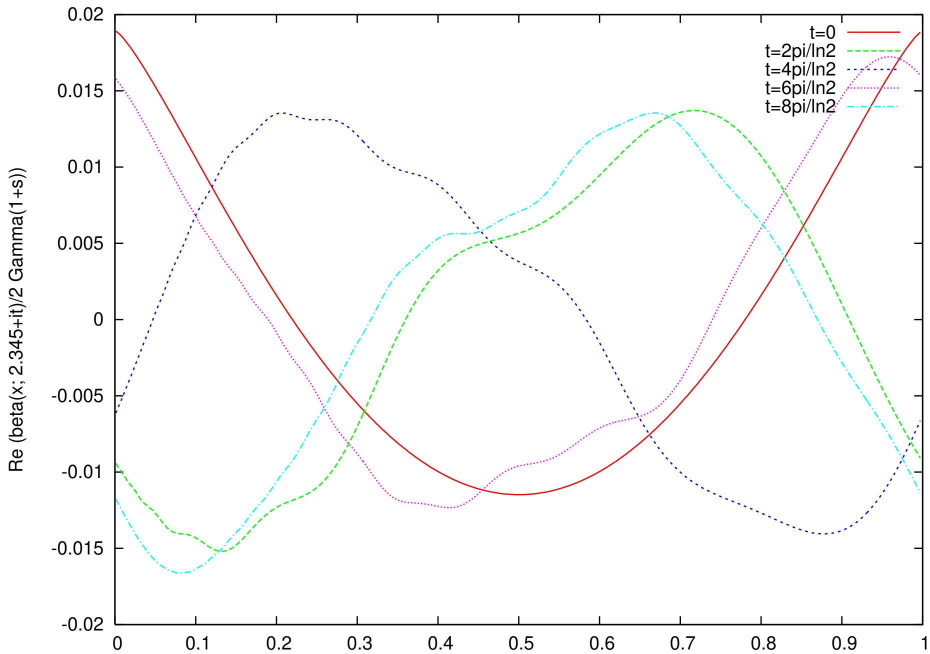
and so even if one starts with $\Re s > 1$, each derivative moves one step closer to the danger zone.

APPENDIX G. APPENDIX: TO-DO LIST

In order to give a proper and complete treatment of the subject, the topics listed below should be presented/reviewed/understood.

- Closure of polynomials – closure suggested by Gaspard[14] is to use the Frechet space of entire holomorphic functions of expoential type less than 2π. The exponential type rules out the sine in the kernel. Claim: the monomials are a basis for this space. Claim: the space is complete. Claim: Bernoulli oper is compact on this space. None of this is totally obvious.
- General question: what is the "largest" space in which Bernoulli oper is compact?
- $\mathscr{L}_B$ is nuclear (Grothendieck) and trace-class thus compact for the polynomials. More generally, under what circumstances is it compact?

FIGURE F.1. Real part of Beta



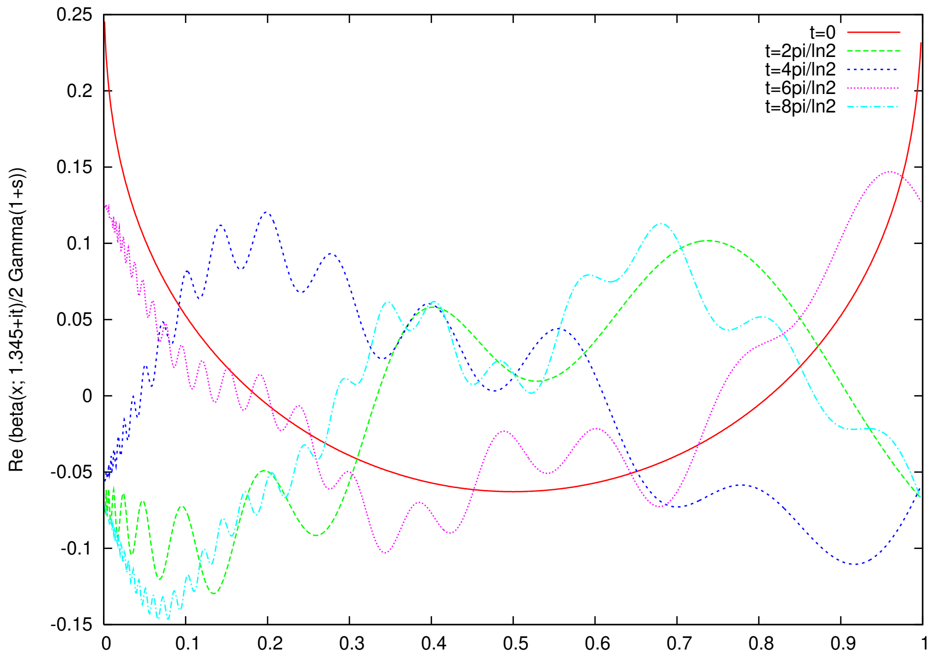
This figure illustrates a family of eigenvectors $\beta(x;s)$ of $\mathscr{L}_B$ having the same eigenvalue 2^{-s}. To be precise, the illustration shows

$$\Re\left(\frac{\beta(x;s)}{2\Gamma(1+s)}\right) = \Re\left((2\pi)^{-s}\operatorname{Li}_s\left(e^{2\pi ix}\right)\right)$$

for values of $s = \sigma + 2\pi in/\ln 2$ for $\sigma = 2.345$ and $n = 0,1,2,3,4$. Note that $\Gamma(1+s)$ gets very small as n gets large, and so the normalization brings them to visually comparable ranges. Although these curves clearly look very lumpy, they are C^∞ for all $x \in (0,1]$ but not at the endpoint $x = 0$. At the endpoint, the derivative becomes divergent after just a few derivatives, where the curves are behaving essentially as x^{s-1}. This pending divergence and the true nature of the lumpiness is made clear in the next graphic. Although these curves appear to be sine-wave-like, it is perhaps more correct to think of them as being Bernoulli-polynomial-like. That is, the general case will look similar to the polynomial $B_{\lfloor \Re s \rfloor}(x)$. Of course, $B_k(x)$ for $k \geq 3$ is very sine-wave like, so the general wave-like nature holds for most values of s. The first curve shown above, for $n = 0$, generally resembles $B_2(x)$, which is a parabola.

- Do we ever have a nuclear space? yes or no?
- Do the general/generic bernoulli shift (defining entorpy Ornstein isomorphism thm.) Do it properly as measure space. define measure preserving dynamical system
- The Bernoulli process in probability theory is one of the simplest Markov processes. Understood as a Markov process, it has a number of generalizations.
- Discuss entropy.
- Discuss the Riesz representation theorem.

FIGURE F.2. Real part of Beta



This figure illustrates a family of eigenvectors $\beta(x;s)$ of $\mathscr{L}_B$ having the same eigenvalue 2^{-s}. To be precise, the illustration shows

$$\Re\left(\frac{\beta(x;s)}{2\Gamma(1+s)}\right) = \Re\left((2\pi)^{-s}\operatorname{Li}_s\left(e^{2\pi ix}\right)\right)$$

for values of $s = \sigma + 2\pi in/\ln 2$ for $\sigma = 1.345$ and $n = 0,1,2,3,4$. The contribution of $\Re x^{s-1} = \cos((\sigma-1)\ln x)$ is more clearly visible in this image; at smaller amplitudes, this ringing makes the curves look lumpy. The impending non-differentiability at $x = 0$ is also manifest.

- Reference the background on symmetry crapola in [29].
- Discuss ergodicity, the ergodic theorem, and how its related to this. Discuss how the Bernoulli map is a “bad” example of ergodicity. Hypothesis: are all transfer operators that are triangular (have a polynomial basis) equivalent to “bad” ergodic sequences? Or are there good (uniformly converging) ergodic sequences that can result from triangular transfer operators?
- Describe C*-algebra of $\mathscr{L}_B$. Explain Toeplitz operator, Toeplitz algebra ... err. but why is this interesting?

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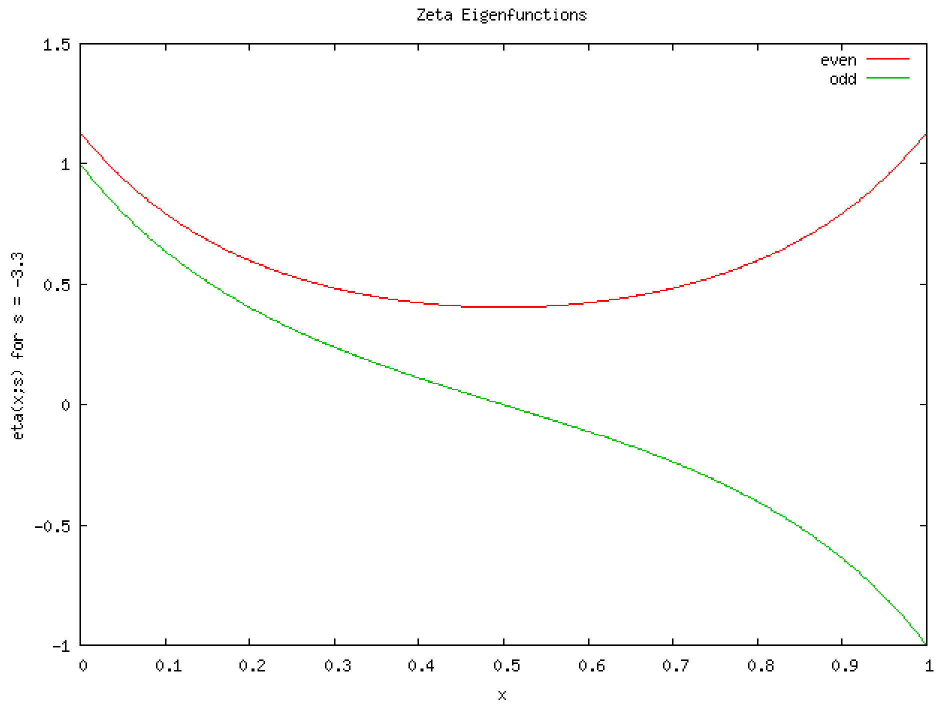
FIGURE F.3. The Non-Singular Part of the Divergent Eigenfunctions



This figure shows

$$\eta_{even}(x;\sigma)=\frac{\cos\pi\sigma/2}{\sigma}\left[\beta(x;\sigma)+\beta(1-x;\sigma)\right]-x^{\sigma-1}-(1-x)^{\sigma-1}$$

and

$$\eta_{odd}(x;\sigma)=\frac{\sin\pi\sigma/2}{\sigma}\left[\beta(x;\sigma)-\beta(1-x;\sigma)\right]-x^{\sigma-1}+(1-x)^{\sigma-1}$$

for a value of $\sigma=-3.3$, corresponding to an eigenvalue of $9.85=2^{3.3}$. Except for the singularity, we see that the finite part of these eigenfunctions is very well behaved.

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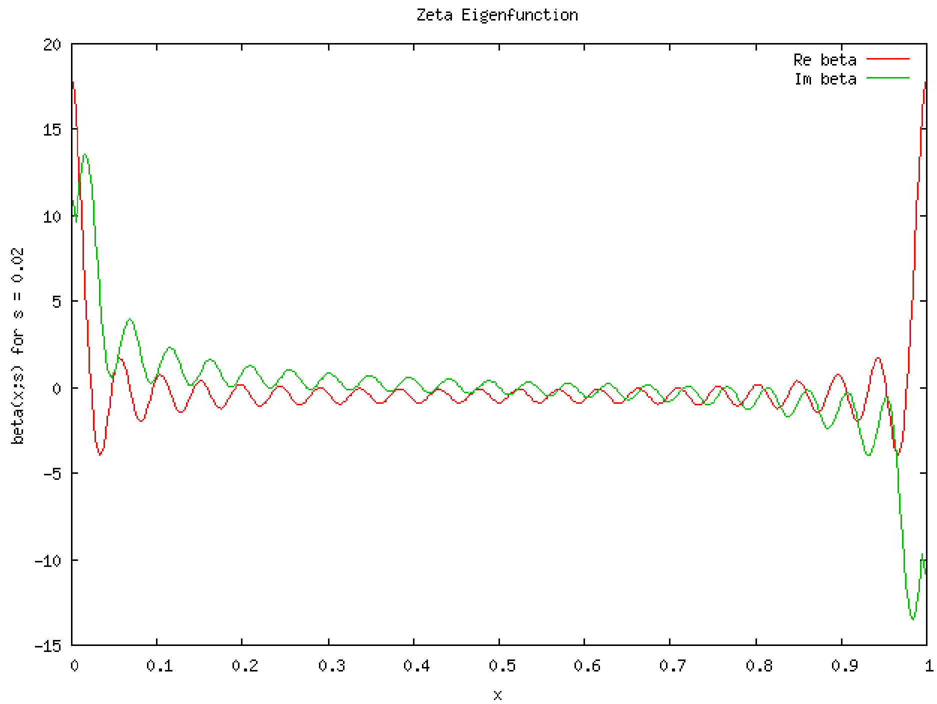
FIGURE F.4. Ringing



This figure shows ringing/Gibbs phenomenon as s approaches zero. In the limit of $s = 0$, we expect the real part of β to approach the trivial eigenfunction $\lim_{s\to 0^+} \Re\beta(x;s) = -B_0(x) = -1$. As this graph shows, the function is indeed trying very desperately to get flat, with not much success. The ringing occurs only at $s = 0$; there is no problem with convergence near larger integers, where $\lim_{s\to n} \Re(-i)^s\beta(x;s) = -B_n(x)$ converges very smoothly and cleanly.

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