A series representation for the Riemann Zeta derived from the Gauss-Kuzmin-Wirsing Operator

Linas Vepstas <linas@linas.org>

2 January 2004 (revised 29 August 2005)

Abstract

A series representation for the Riemann zeta function in terms of the falling Pochhammer symbol is derived from the polynomial representation of the Gauss-Kuzmin-Wirsing (GKW) operator.

1 Introduction

The Gauss-Kuzmin-Wirsing operator [Kuz28, Wir74, Ba78a] occurs in the theory of continued fraction representations of the real numbers. The zeroth eigenvalue of this operator is related to the Gauss-Kuzmin distribution [ref], giving the likelihood of the occurrence of an integer in the continued-fraction expansion of an "arbitrary" real number. The GKW operator is interesting in other ways, notably in its relationship to the Minkowski Question Mark function, and thus its symmetry under the action of a monoid sub-semi-group of the modular group $SL(2,\mathbb{Z})$. Except for the zeroth eigenvector, there is no known closed-form solution of the GKW operator, although the eigenvalues may be computed relatively easily through standard matrix diagonalization techniques.

In this paper, a relationship to the Riemann zeta function [Ed74] is noted, allowing the easy derivation of a series expansion of the zeta function in terms of the Pochhammer symbols (the falling factorials), or equivalently the binomial coefficients. Specifically, the expansion given is

$$\zeta(s) = \frac{s}{s-1} - s\sum_{n=0}^{\infty}(-)^n \binom{s-1}{n} t_n \tag{1}$$

Here, the constants t_n play a role analogous to the Stieltjes constants, and can be given in a simple, finite-term closed form, involving the Riemann zeta function at integer values and the Euler-Mascheroni constant.

The t_n may be used to express the Stieltjes constants as a sum involving the Stirling numbers. The t_n are very well behaved, becoming small quite rapidly. Some exact expressions can be derived for various sums involving these constants. A numerical exploration of their values is also performed, from which the limiting behavior is conjectured.

The first section defines the GKW operator in simple terms, as the transfer operator of the Gauss map. The second section briefly reviews some of the properties of the GKW operator, and the third section provides several polynomial representations. The fourth section obtains the series expansion for the zeta. The fifth section explores the values of the constants t_n.

2 The Gauss-Kuzmin-Wirsing Operator

Successive terms of a continued fraction fraction expansion for a real number x may be obtained by iterating on the function

$$h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \tag{2}$$

This function maps the unit interval onto itself, and is sometimes called the Gauss map. It is connected to the Riemann zeta function by a Mellin transform:

$$\zeta(s) = \frac{1}{s-1} - s\int_0^1 h(x)\, x^{s-1} dx \tag{3}$$

In the theory of iterated functions, the behavior of a map may be studied by means of the “transfer operator”, sometimes called the Frobenius-Perron operator of a map [Rue94]. Given a map of the unit interval onto itself, the transfer operator describes how a distribution on the unit interval behaves under the action of the map, thus giving an broader view of iteration. In particular, the transfer operator can act as a bridge between the fractal properties typically seen when iterating a function, and analytic descriptions of the same process. This is because transfer operators can be represented both on the space of smooth, differentiable functions, as well as spaces of non-differentiable functions.

By using a set of smooth, differentiable, orthonormal basis functions on the unit interval, such as orthogonal polynomials, the matrix elements of a transfer operator can often be explicitly computed, and the operator itself can be diagonalized, using standard techniques from the theory of Hilbert spaces and functional analysis.

There are a variety of equivalent definitions for the transfer operator, of varying degrees of sophistication. One simple definition is

$$\left[U_h \rho\right](x) = \int dy\, \delta\left(x - h(y)\right)\, \rho(y) \tag{4}$$

Here, h is the function being iterated, while ρ is some density on which the transfer operator U_h acts. The subscript on U_h is used to emphasize that the transfer operator is associated with the function being iterated. In this expression, δ is understood to be the Dirac delta function. As can be seen, as iteration maps one point to another, the transfer operator describes how the density is mapped.

A more abstract definition may be given. Let $h : X \to X$ be a map from an arbitrary set X onto itself. Let $\rho : X \to \mathbb{C}$ be a valuation of that set on the complex numbers $\mathbb{C}$, and let $\mathcal{R}$ be the space of all such valuations ρ. Then the transfer operator U_h is a linear functional on the space $\mathcal{R}$, that is, $U_h : \mathcal{R} \to \mathcal{R}$, acting as

$$[U_h\rho](x) = \sum_{y \in h^{-1}(x)} \rho(y) \tag{5}$$

Rather than studying U_h acting on the whole space $\mathcal{R}$, it is often insightful to represent U_h acting only on a subspace. For example, when X is the unit interval, one may consider only the space of (finite) polynomials, or the space of analytic functions, or the space of square-integrable functions, etc.

For the Gauss map, the transfer operator is known as the Gauss-Kuzmin-Wirsing (GKW) operator, and has the representation [Khi35]

$$[U_h\rho](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} \rho\left(\frac{1}{n+x}\right) \tag{6}$$

The definition 4 can be loosely viewed as a kind of change-of-variable. This change-of-variable can be performed on equation 3, the order of integration exchanged, and the Gauss map replaced by its transfer operator. One obtains the operator equation

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x \left[U_h x^{s-1}\right] \tag{7}$$

Although the above equation 7 is easy enough to derive, a proper, rigorous examination of its validity requires a deep study of topics in functional analysis, topics outside the scope of this paper. Wirsing does present some arguments concerning the existence and analyticity of the eigenvectors of this operator; however, the arguments are specific to this operator and do not draw on any general theory [Wir74].

This expression of the Riemann zeta function in terms of the GKW operator motivates a deeper study to the GKW operator, and of transfer operators in general. Most curiously, continued fractions have a variety of interesting connections to the modular group $SL(2,\mathbb{Z})$, and indeed, the modular group can be viewed as a symmetry group of the dyadic tree representation of the unit interval afforded by both the binary numbers and the Farey numbers. Thus, one is lead to ask if the Riemann Hypothesis (RH) can be deduced from a set of symmetries/invariances of $SL(2,\mathbb{Z})$ acting on the unit interval, and, in particular, the symmetries of operators that commute with the GKW operator.

A complete analysis of the GKW operator [Ba78a] has never been given, and appears to be difficult at many levels. The operator has one well-known eigenvector, $\rho(x) = 1/(1+x)$, called the Gauss distribution, as it was known to Gauss. It corresponds to the unit eigenvalue. A closed form for the other eigenvectors is not known. The next eigenvalue has the approximate value $\lambda_1 \approx 0.3036630029...$ and is known as the Gauss-Kuzmin-Wirsing constant; it gives the rate of (exponential) convergence of distributions of continued fraction expansions to the Gauss distribution.

The author has attempted a computer-guided search for a closed-form solution in terms of a simple linear combination of products of common special functions, including the factorial, the digamma, and Bessel functions. No solution was found; it is not clear that the eigenvectors have any simple solution in terms of common special functions.

XXX Todo: Give the Riemann Hypothesis as a vector equation.

3 Assorted Algebraic Identities

The action of the GKW operator can be computed explicitly for a variety of simple functions. This section lists an assortment of random algebraic results, none particularly deep, although many are suggestive in various ways. These are listed here mostly for the sake of completeness.

Adjacent terms in the series can be made to cancel by shifting the series by one:

$$[U_h f](x) - [U_h f](x+1) = \frac{1}{(1+x)^2} f\left(\frac{1}{1+x}\right) \tag{8}$$

which holds for any function $f(x)$. Thus, if $\rho(x)$ is an eigenvector, so that $U_h\rho = \lambda\rho$, then it would also solve

$$\frac{1}{(1+x)^2}\rho\left(\frac{1}{1+x}\right) = \lambda\left(\rho(x) - \rho(x+1)\right) \tag{9}$$

This can be solved easily to get the zeroth eigenvector

$$\rho_0(x) = \frac{1}{\ln 2}\,\frac{1}{1+x} \tag{10}$$

which satisfies $[U_h\rho_0](x) = \rho_0(x)$ and the normalization is given by requiring

$$\int_0^1 \rho_0(x)\,dx = 1 \tag{11}$$

A reflection identity is given by $f(x) = 1-(1+x)^{-2}$ which satisfies $U_h f = 1-f$.

A hint of the relation to the modular group is given by simple identities involving the Minkowski Question Mark function $?(x)$:

$$U_h[(1+?(x))/(1+x)^2] = 1-?(x) \tag{12}$$

The Minkowski Question Mark function is an isomorphism of dyadic trees; specifically the binary tree that represents the real numbers as a binary expansion, and the Stern-Brocot or Farey tree, the binary tree that represents the real numbers by the rational numbers. In this context, its important to keep in mind that the modular group $SL(2,\mathbb{Z})$ is the group of symmetries or hyperbolic rotations of binary trees, and that the Question Mark enjoys a set of fractal self-similarities under the action of the modular group. Another question mark identity is

$$U_h[?(x)\,x^{-2}] = 2-?(x) \tag{13}$$

Additional identities may be constructed in this vein, for example:

$$U_h \left[?(x) \left(\frac{1}{(1+x)^2} - 2 \right) \right] = \frac{?(x)-2}{(1+x)^2} \tag{14}$$

but these types of exercises do not seem to lead to any sort of obviously worthwhile recurrence relations; a deeper analysis of symmetries is required.

Acting on the monomial, one gets

$$\left[U_h x^s \right](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^{s+2}} = \zeta_H(s+2, 1+x) \tag{15}$$

where $\zeta_H(s,q)$ is the Hurwitz zeta function. For positive integer $s=k$, one has that $\zeta_H(k+1,x) = (-1)^{k+1}\psi^{(k)}(x)/k!$ where $\psi^{(k)}(x)$ is the polygamma function, the k'th derivative of the gamma function. For negative integers, one gets the Bernoulli polynomials: $\zeta_H(-n,x) = -B_{n+1}(x)/(n+1)$. As a variation, one has curiosities such as $f(x) = (1+ax)^2$ giving $U_h f = \psi^{(1)}(1+x+a)$ and $f(x) = (1+nx)^{-2} - 1$ giving $U_h f = -\sum_{k=1}^{n}(x+k)^{-2}$ (this last being a finite sum).

The relatively simple expressions involving the Hurwitz zeta and the Minkowski question mark function again strongly hint that the modular group is deeply involved in the correct theory of the GKW operator. For example, the Bernoulli map $b(x) = 2x - \lfloor 2x \rfloor$, when iterated, gives the binary digits of the real number x. The transfer operator of the Bernoulli map is exactly solvable: the eigenfunctions are the Bernoulli polynomials, and more generally, the Hurwitz zeta function.

In closing, one teasing hint of the relationship between period-doubling and the GKW is this curious representation of the zeroth eigenvector as a sum over progressively smaller intervals:

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{2}{x+n} - \frac{1}{x+n+1} \right] \tag{16}$$

4 Polynomial Representation

The search for analytic solutions to the GKW operator begins with a specification of the operator in a polynomial basis. To that end, a natural first choice is to consider a representation in terms of the monomials x^n or equivalently, a Taylor expansion about $x=0$. This proves to be an unfortunate choice, as will become clear; The GKW operator appears to have some sort of singularity associated with the point $x=0$. Writing $U_h f = g$ and substituting a Taylor's expansion for f and g, one finds

$$\frac{g^{(m)}(0)}{m!} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (-)^m \frac{(k+m+1)!}{m!\,(k+1)!} \zeta(k+m+2) \tag{17}$$

The operator matrix elements can be immediately [May91] read off to be

$$\left[U_h \right]_{mk} = (-1)^m \begin{pmatrix} k+m+1 \\ m \end{pmatrix} \zeta(k+m+2) \tag{18}$$

where the factorials are replaced by the binomial coefficient that they form. Unfortunately, this is a poorly conditioned matrix, with a rapidly increasing matrix elements and a trace that does not converge. Progress may be made by applying a regulator and using Levin-type sequence acceleration techniques[ref]. This leads to a number of curious identities, some of which were listed previously. However, the difficulty of working with divergent sums seems to outweigh any advantages given by the relatively simple form of the matrix elements.

Presuming that there is some sort of singularity at $x = 0$, one may try expansions at different locations. Thus, consider $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\,(x-a)^n/n!$ and $g(x)$ likewise expanded about $x = b$. With this expansion, the operator relation $U_h f = g$ becomes

$$\frac{g^{(m)}(b)}{m!} = \sum_{n=0}^{\infty} U_{mn}^{(b,a)} \; \frac{f^{(n)}(a)}{n!} \tag{19}$$

Without much difficulty, one discovers that the matrix elements are given by

$$U_{mn}^{(b,a)} = (-1)^m \sum_{k=0}^{n} (-a)^{n-k} \binom{n}{k} \binom{k+m+1}{m} \zeta_H(k+m+2, 1+b) \tag{20}$$

where $\zeta_H(s,q)$ is the Hurwitz zeta function:

$$\zeta_H(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \tag{21}$$

Substituting $a = b = 1/2$, one obtains the well-known expansion of [Br03], which is

$$U_{mn}^{(1/2,1/2)} = (-1)^m \sum_{k=0}^{n} \left(\frac{-1}{2}\right)^{n-k} \binom{n}{k} \binom{k+m+1}{m} \left[2^{m+k+2}\left(\zeta(k+m+2)-1\right) - \zeta(k+m+2)\right] \tag{22}$$

A simpler expression is obtained by expanding about $a = b = 1$, and the matrix is rather well-conditioned and easier to work with. It is:

$$G_{mn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+m+1}{m} \left[\zeta(k+m+2) - 1\right] \tag{23}$$

which satisfies

$$(-)^m \frac{g^{(m)}(1)}{m!} = \sum_{n=0}^{\infty} G_{mn} (-)^n \frac{f^{(n)}(1)}{n!} \tag{24}$$

The above will be the most convenient for expressing the Riemann zeta, and will be used in the next section.

All of these expressions for the matrix elements for the GKW operator have a common form. It consists of two summations: the outer summation, and the summation defining the Hurwitz zeta function. Pulling out this second summation, one finds terms consisting of a series of polynomials, which are most simply expressed in terms of Gauss' hypergeometric series:

$$\Gamma_{mn}(x) \equiv (m+1)\,{}_2F_1\left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array};x\right] = \sum_{k=0}^{n}\binom{n}{k}\binom{k+m+1}{m}(-x)^k \tag{25}$$

These have a curious superficial resemblance to the shifted Legendre polynomial

$$\widetilde{P}_n(x) \equiv \sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{n}(-x)^k \tag{26}$$

Switching the order of summation in equation 20 gives the following:

$$U_{mn}^{(b,a)} = (-1)^{m+n}a^n\sum_{j=0}^{\infty}\frac{1}{(j+1+b)^{m+2}}\,{}_2F_1\left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array};\frac{-1}{a(j+1+b)}\right] \tag{27}$$

Oddly, this appears to be in the form of a slightly generalized form of the GKW operator, the Ruelle-Mayer operator [May91],

$$\left[U^{(s)}f\right](x) \equiv \sum_{n=1}^{\infty}\frac{1}{(n+x)^s}f\left(\frac{1}{n+x}\right) \tag{28}$$

where s is taken to be $s = m+2$ and $f =_2 F_1$. The general appearance and the matrix elements of the Ruelle-Mayer operator are only a slight variation of those for the GKW; and, for completeness, these are:

$$\left[U^{(s)}\right]_{mn}^{(b,a)} = (-1)^m\sum_{k=0}^{n}(-a)^{n-k}\binom{n}{k}\binom{m+k+s-1}{m}\zeta_H(m+k+s,1+b) \tag{29}$$

The corresponding hypergeometric identity that comes into play is

$$\sum_{k=0}^{n}(-x)^k\binom{n}{k}\binom{m+k+s-1}{m} = \binom{m+s-1}{m}\,{}_2F_1\left[\begin{array}{cc} -n & m+s \\ & s \end{array};x\right] \tag{30}$$

As a final note, recall that the Hurwitz zeta may be expressed as the polygamma function for integer arguments, and that the polygamma functions are the chain of logarithmic derivatives of the gamma function. Thus, one may also expresses the matrix elements of U in the curious form

$$U_{mn}^{(b,a)} = \frac{(-a)^{n+1}}{m!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{a}\right)^{k+1}\frac{1}{(k+1)!}\frac{d^{k+1}}{dx^{k+1}}\psi^{(m)}(1+b) \tag{31}$$

Here, the curious operator making an appearance is

$$\left[P_{n,y}f\right](x) = \sum_{k=0}^{n}(-y)^k\binom{n}{k}\frac{f^{(k)}(x)}{k!} \tag{32}$$

where $f^{(k)}(x)$ is the k'th derivative of f at x. The operator $P_{n,y}$is upper-triangular, with all eigenvalues equal to 1, and all eigenvectors being polynomials (or analytic series for n not an integer).

5 The Riemann Zeta and Stieltjes Constants

Inserting the representation 23 into the integral expression 7 for the Riemann zeta gives

$$\zeta(s) = \frac{s}{s-1} - s\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{G_{mn}(-)^n}{(m+1)(m+2)}\left(\begin{array}{c} s-1 \\ n \end{array}\right) \tag{33}$$

This expression may be considerably simplified. Several of the sums appearing in equation 33 may be performed explicitly. Using $t_n$to denote the intermediate sum, one may write

$$\zeta(s) = \frac{s}{s-1} - s\sum_{n=0}^{\infty}(-)^n\left(\begin{array}{c} s-1 \\ n \end{array}\right) t_n \tag{34}$$

It is not to difficult to verify that the $t_n$take the form

$$\begin{aligned} t_n &= \sum_{m=0}^{\infty}\frac{G_{mn}}{(m+1)(m+2)} \\ &= 1-\gamma+\sum_{k=1}^{n}(-)^k\left(\begin{array}{c} n \\ k \end{array}\right)\left[\frac{1}{k}-\frac{\zeta(k+1)}{k+1}\right] \end{aligned} \tag{35}$$

Here, $\gamma = 0.577...$ is the Euler-Mascheroni Constant. For large n, one finds that $t_n \to 1/2(n+1)$, motivating the definition of

$$a_n = t_n - \frac{1}{2(n+1)} \tag{36}$$

so that the Riemann zeta may be written as

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\sum_{n=0}^{\infty}(-)^n\left(\begin{array}{c} s-1 \\ n \end{array}\right) a_n \tag{37}$$

Writing the binomial coefficient as $\left(\begin{array}{c} s-1 \\ n \end{array}\right) = (s-1)_n/n!$ where $(x)_n$ is the falling Pochhammer symbol, its clear that the a_n play the analogue of the Stieltjes constants for this kind of Umbral, "divided differences" Newton-type equation. Unlike the Stieltjes constants, the a_n have a simple, finite expression. For example, the first few may be written as

$$a_0 = \frac{1}{2} - \gamma \tag{38}$$

$$a_1 = \frac{\zeta(2)}{2} - \gamma - \frac{1}{4} \tag{39}$$

$$a_2 = \zeta(2) - \frac{\zeta(3)+2}{3} - \gamma \tag{40}$$

$$a_3 = \frac{3}{2}\zeta(2) - \zeta(3) + \frac{\zeta(4)}{4} - \frac{23}{24} - \gamma \tag{41}$$

and so on. Numerically, the a_n are small, and seem to be bounded and oscillatory. A detailed numerical analysis of these values is given in the next section.

Identities useful in the course of the manipulations needed to derive the above include

$$H_n = \sum_{m=1}^{n} \frac{1}{m} = -\sum_{k=1}^{n} \frac{(-1)^k}{k} \left(\begin{array}{c} n \\ k \end{array} \right) \tag{42}$$

and

$$-\sum_{k=0}^{n} \frac{(-x)^{k+1}}{k+1} \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{1-(1-x)^{n+1}}{n+1} \tag{43}$$

This last identity is useful for partially resumming equation 35 to improve convergence in numerical calculations.

The $a_n$appear naturally in the Taylor's expansion for the Gamma function, and so one finds, with little difficulty, the generating function

$$\begin{aligned} \alpha(z) &= \sum_{n=0}^{\infty} a_n z^n \\ &= \frac{1}{1-z} + \frac{\ln(1-z)}{z} \left(\frac{1}{1-z} - \frac{1}{2} \right) + \frac{1}{z} \ln \Gamma \left(\frac{1}{1-z} \right) \end{aligned} \tag{44}$$

The above may be obtained by the straightforward application of equation 6.1.33 from Abramowitz and Stegun [AS64]. This representation shows a complicated structure. There is a cut (from the logarithm) extending to the right for $1 < z$. In the cut there are poles (from the gamma function) at $-n = 1/(1-z)$, that is, at $z = 1 + 1/n$, accumulating to at $z = 1$. There appears to be a simple zero at $z = \infty$. The radius of convergence of the series involving the $a_n$is one. As will be seen in the next section, the a_n are oscillatory and exponentially decreasing, and so the sum at $z = 1$is convergent. One has:

$$\sum_{n=0}^{\infty} a_n = \ln\sqrt{2\pi} - 1 = -0.081061466795327... \tag{45}$$

which can be obtained from Sterling's asymptotic expansion for the gamma function, and

$$\sum_{n=0}^{\infty} a_n 2^{-n} = 2 - 3\ln 2 = -0.079441541679836... \tag{46}$$

An exponential generating function is given by

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} = \frac{1}{2z} + e^z \left[1 - \gamma - \mathrm{Ein}(z) - \frac{1}{z} \left(\frac{1}{2} + \sum_{k=2}^{\infty} \frac{(-z)^k}{k!} \zeta(k) \right) \right] \tag{47}$$

where $\mathrm{Ein}(z)$ is the entire exponential integral:

$$\mathrm{Ein}(z) = -\sum_{k=1}^{\infty} \frac{1}{k} \frac{(-z)^k}{k!} = e^{-z} \sum_{n=1}^{\infty} H_n \frac{z^n}{n!} \tag{48}$$

and H_n are the harmonic numbers.

Its equally curious that other variations on the expression given in 37 are even more trivial, namely, one has

$$\sum_{n=0}^{\infty} (-)^n \binom{s-1}{n} x^n = \sum_{n=0}^{\infty} (-)^n (s-1)_n \frac{x^n}{n!} = (1-x)^{s-1} \tag{49}$$

where $(s)_n = s(s-1)(s-2)...(s-n+1)$ is the falling factorial. One is left to wonder what the function

$$\mu(s;x) = \frac{s}{s-1} - \frac{1}{2} - s\sum_{n=0}^{\infty} (-)^n \binom{s-1}{n} a_n x^n \tag{50}$$

might be like; it is presumably related to the Polylogarithm (Jonquiere's function) XXXX Do the scratching needed to get the relationship nailed down. Ugh. The general idea of replacing power series by series in rising or falling Pochhammer symbols and then exploring the curious relationships that result is referred to as Umbral Calculus; it seems that a number of interesting relationships can be obtained in this way.

6 Numerical Analysis

The equation 36 is very conducive to numerical analysis. The values of the zeta function for small integers are known to millions of digits; Plouffe has given a series of very rapidly converging sums for the larger integers[ref]. Thus, it becomes possible to explore the behavior of the a_n up to $n \sim 500$ with only a moderate investment in coding and computer time. Note, however, that in order to get up to these ranges, computations must be performed keeping hundreds of decimal places of precision, due to the very large values that the binomial coefficient can take. In the following, calculations were performed using the GNU Multiple Precision Arithmetic Library [GMP].

The table 1 shows the numerical values of the first few of these constants. It appears that the $a_n$are oscillatory, as shown in the figure 1. The values are also very rapidly decreasing, so that to the first order, that $|a_n| \sim \exp\left(-4\sqrt{n+1}\right)$. This exponential decrease is vaguely reminiscent of an exact result from Babenko [Ba78b], and it is possible that Babenko's analysis might be applied to yeild a more precise statement of the results of this section.

The author attempted a numeric fit to the oscillatory behavior of the function. An excellent fit for the $k'th$ zero is provided by

$$q(k) = 1.970 + \frac{17\pi}{16} k + \frac{\pi}{4} k^2 \tag{51}$$

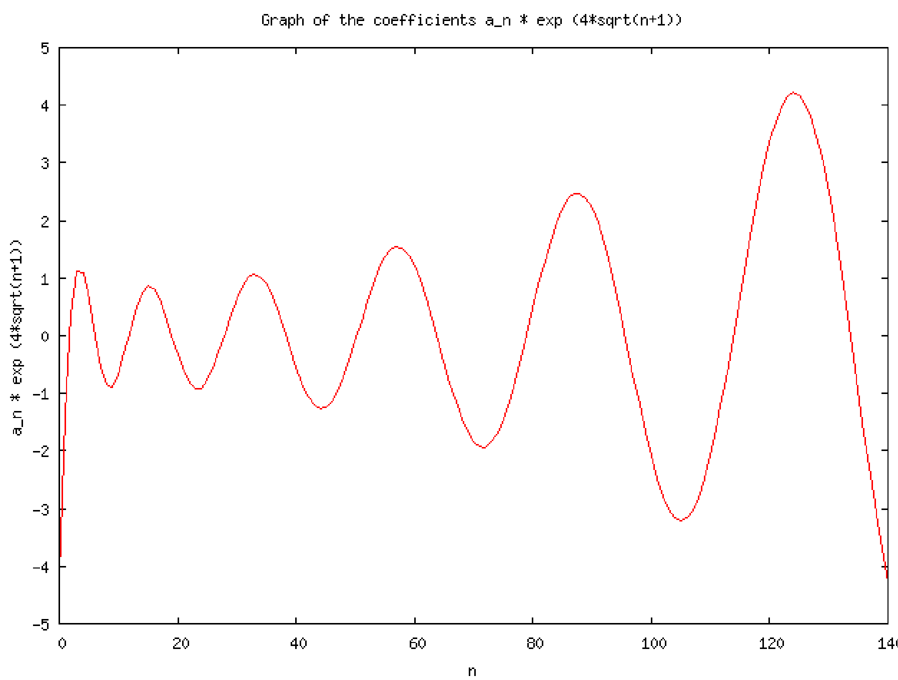
which appears to fit the zero-crossings to better than a few parts per thousand, at least for the larger values of k. Note that the value 17/16 is not meant to imply that this value

Table 1: Some values of a_n

n	a_n
0	-0.077215664901532851...
1	-0.0047486314774196408...
2	0.00036610089349548089...
3	0.00037600730566372326...
4	0.00014301182486231440...
5	3.3997818936021684...e-5
6	-4.8324221220657863...e-7
7	-6.7778497812918602..e-6

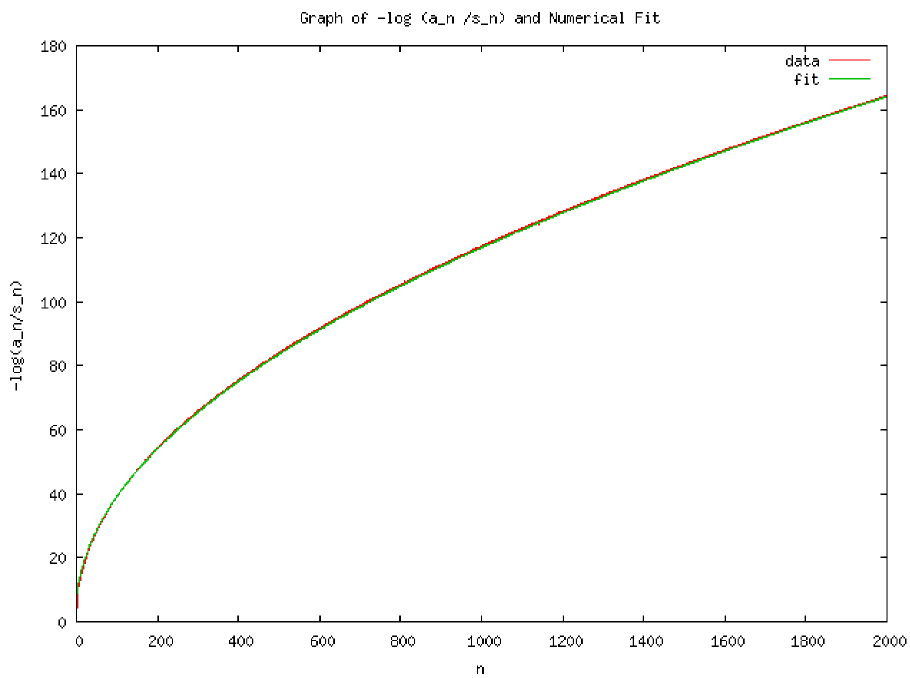
This table shows the first few values of a_n. These may be easily calculated to high precision if desired. As is immediately apparent, these get small quickly.

Figure 1: Graph of the first few values of a_n



This figure shows a graph of the first 140 values of a_n normalized by a factor of $\exp -4\sqrt{n+1}$. Oscillation is clearly visible; the period of oscillation is slowly increasing.

Figure 2: Graph of the log of the amplitude a_n/s_n



This figure shows a graph of $-\log(a_n/s_n)$ out through $n \sim 2000$. It is essentially a graph of the log of the amplitude of the oscillations of a_n. Note the absence of any 'blips' that would occur if the sinusoidal fit to the data was poor; the sinusoidal curve s_n models the data remarkably well. This figure also illustrates a fit to the amplitude. Specifically, the fit curve is given by $3+3.6\sqrt{n+1}$. Note that this fit fails badly for values of $n \lesssim 20$ and that for larger n, the fit is only good to $3+3.6\sqrt{n+1}+\log(a_n/s_n) \simeq 0 \pm 0.5$ on average.

is truly exact; numerically, the value 1.0625 seems to be the ideal fit, and it is written here as 17/16 in order to be suggestive. This equation is trivially invertible to give the oscillatory behavior of the a_n; it is given by

$$s_n = \sin \pi \left(\frac{-1}{16} + \sqrt{\frac{289}{256} + 4\,\frac{(n-1.97)}{\pi}} \right) \tag{52}$$

To demonstrate the quality of this fit, the graph 2 shows a_n divided by s_n.

Asymptotically, the amplitude of the a_n appears to be given by

$$a_n \simeq s_n \exp - \left(3 + 3.6\sqrt{n+1} \right) \tag{53}$$

However, a more precise fit is strangely difficult. An attempt to numerically fit the data with an expression of the form

$$a_n \simeq s_n \exp - \left(b + (c+dn)^{\varepsilon}\right) \tag{54}$$

results in fit parameters of $b = 3.9 \pm 1.5$ and $c = -50 \pm 100$ and $d = 12.87 \pm 0.15$ and $\varepsilon = 0.5 \pm 0.007$. This fit is not very pleasing; in particular, the true asymptotic behavior doesn't seem to set in until n is larger than several hundred, which is unexpectedly high.

7 A Related Function

The $a_n$may be further explored by generalizing them to be a function on the complex plane, instead of an integer. One such "obvious" and reasonable generalization is

$$a(s) = 1 - \gamma - \frac{1}{2(s+1)} + \sum_{k=1}^{\infty} (-)^k \left(\begin{array}{c} s \\ k \end{array} \right) \left[\frac{1}{k} - \frac{\zeta(k+1)}{k+1} \right] \tag{55}$$

By construction, this function interpolates the previous series, so that $a(n) = a_n$ for non-negative integers n. This function has a pole at $s = -1$, and this series expansion is convergent only for $\Re s > -1$. The figure 3 illustrates $a(s)$ on the imaginary axis; the graphic 4 illustrates the phase of $a(s)$ in the upper-right complex plane. A numerical exploration of this function seems to indicate that the only zeros of this function occur on the real number line.

Of some utility are two identities, valid for $\Re s > -1$:

$$-\sum_{k=0}^{\infty} \left(\begin{array}{c} s \\ k \end{array} \right) \frac{(-y)^{k+1}}{k+1} = \frac{1-(1-y)^{s+1}}{s+1} \tag{56}$$

and

$$\sum_{k=1}^{\infty} \left(\begin{array}{c} s \\ k \end{array} \right) \frac{(a-1)^k}{k} = \int_a^1 \frac{1-x^s}{1-x} dx \tag{57}$$

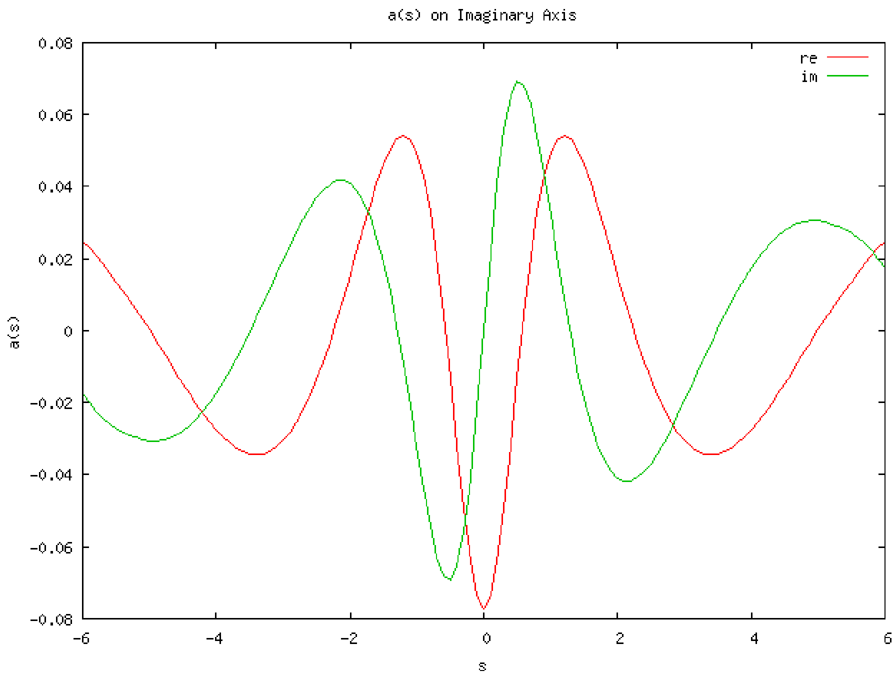
For $a = 0$, the last integral becomes the digamma function; see equation 6.3.22 of [AS64].

8 Relation to the Stieltjes constants

The a_n are can be used to express the Stieltjes constants and vice-versa by re-expressing the binomial coefficient with a power series, making use of Stirling Numbers. That is, the polynomial expression is given by

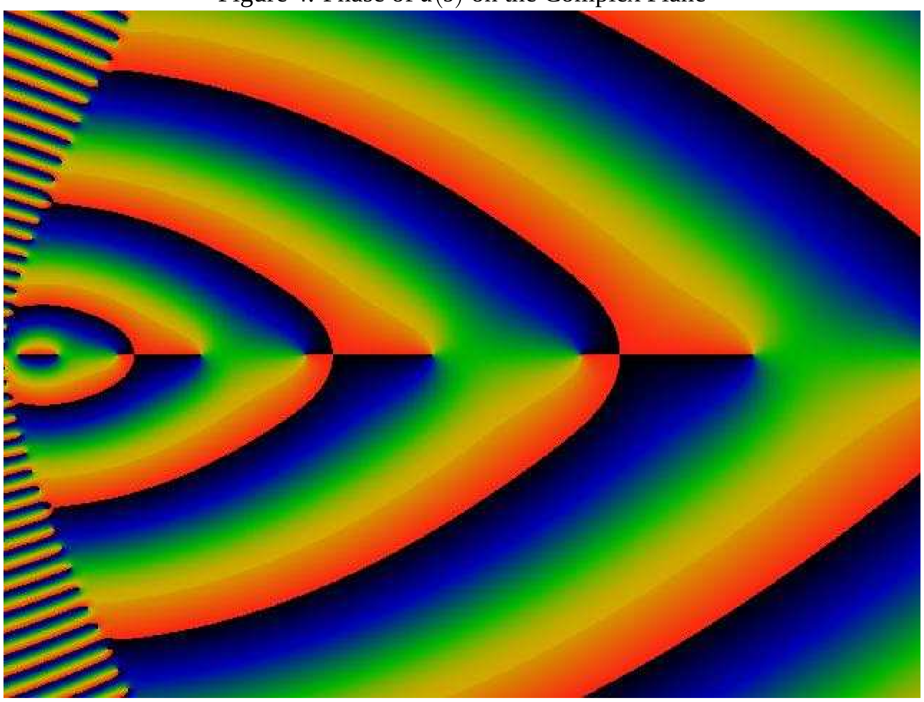
$$\left(\begin{array}{c} s-1 \\ n \end{array} \right) = \frac{(s-1)_n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] (s-1)^k \tag{58}$$

Figure 3: $a(s)$ on the Imaginary Axis



This figure shows the real and imaginary parts of $a(s)$ along the imaginary axis. The function appears to be oscillatory. Note that the series expansion converges only with great difficulty as one goes further out on the imaginary axis.

Figure 4: Phase of $a(s)$ on the Complex Plane



This figure shows the phase of $a(s)$ on the right half of the complex plane, running along the real-axis from -2 to +62 and the imaginary axis from -24 to about +24. The ribbing on the left is entirely due to numerical errors, as the sums converge only with great difficulty. The color scheme is such that black corresponds to $-\pi$, green to 0 and red to $+\pi$, on a smooth scale. Thus, the red-to-black color discontinuities, correspond to a phase change of 2π. Note that these terminate on zeros or poles; in this case, it appears that there is only one pole, at $s = -1$, and all others are zeros. In particular, it appears that all of the zeros occur on the real number line.

where $\left[\begin{array}{c} n \\ k \end{array} \right]$ is the Stirling Number of the First Kind. Substituting in the above, and comparing to the standard definition of the Stieltjes constants

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \gamma_n (s-1)^n \tag{59}$$

shows that $\gamma_0 = 1/2 - a_0 = \gamma$ and

$$\gamma_k = -ka_{k-1} + (-)^k k! \sum_{n=k}^{\infty} (-)^n \frac{a_n}{n!} \left(\left[\begin{array}{c} n \\ k \end{array} \right] + \left[\begin{array}{c} n \\ k-1 \end{array} \right] \right) \tag{60}$$

Note that the Stirling Numbers can be written as a sum over a product of harmonic numbers. That is,

$$\left[\begin{array}{c} n \\ k \end{array} \right] = (-)^{k-n} \frac{(n-1)!}{(k-1)!} w(n, k-1) \tag{61}$$

where $w(n,0) = 1$ and

$$w(n,k) = \sum_{m=0}^{k-1} \frac{\Gamma(1-k+m)}{\Gamma(1-k)} H_{n-1}^{(m+1)} w(n, k-1-m) \tag{62}$$

and the Harmonic numbers $H_n^{(m)}$ are given by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \tag{63}$$

This finally allows us to write

$$\gamma_k = -ka_k + k \sum_{n=k}^{\infty} \frac{a_n}{n} \left(w(n, k-1) - (k-1) w(n, k-2) \right) \tag{64}$$

and so the factorial factors cancel, leaving only the sum over the crazy product of harmonics. XXX todo show some of the values of w, esp. along the diagonal. XXX

While on the topic of Umbral relations, the application of Newton's divided differences to the Riemann zeta function leads to the curious function

$$Q(z) = \sum_{n=0}^{\infty} \left(\begin{array}{c} n+1 \\ z-1 \end{array} \right) [\zeta(n+2) - 1] \tag{65}$$

which has the curious properties that $Q(n) = \zeta(n)$ for all integers $n \geq 2$. The pole is absent: $Q(1) = 1$ and $Q(n) = 0$ $\forall$integers $n \leq 0$. The analytic structure of $Q(z)$ is unclear.

9 Conclusions

In the above, an expansion for the Riemann Zeta in terms of the falling factorial was stumbled upon. The falling factorial is a polynomial in s and takes part in many interesting identities that suggest further exploration. In fact, the polynomial $(s-1)_n$ as well as $(s-1)^n$ both fall into a class of functions with share many common properties with respect to differentiation, exponentiation, translation and the like, and are known as Sheffer sequences. It seems that it could be interesting to provide a description of the Riemann zeta expanded in Sheffer sequences; that is, to find the w_n, and provide a general description of the expansion

$$\zeta(s) = \frac{s}{s-1} - s\sum_{n=0}^{\infty}(-)^n w_n p_n(s-1) \tag{66}$$

where the polynomials $p_n(x)$ form a Sheffer sequence. This undertaking is interesting because there may be one particular Sheffer sequence for which the "generalized Stieltjes constants" w_n take on a particularly simple or interesting form. This is already suggested by the presentation given in this text, where the expression of the a_n is considerably simpler than most presentations of the traditional Stieltjes constants. Such work might be made doubly interesting by the fact the theory of Sheffer sequences often shows up in the theory of lattice paths and tilings, whereas the representation of the real numbers as p-adics or rationals is essentially a kind of lattice representation (having a modular group symmetry).

The GKW operator also has an overt relationship to continued fractions, and thus to the modular group. it would certainly be interesting to explore how the group structure manifests itself in the GKW operator, and, in turn, what this might imply for the structure of the Riemann zeta.

References

[Gas92] P. Gaspard, *r-adic one-dimensional maps and the Euler summation formula*, 1992, Journal of Physics A: Mathematical and General, vol. 25, L483-485.

[Iso03] Stefano Isola, *On the Spectrum of Farey and Gauss Maps*, preprint, undated (2004 or earlier)

[Apo76] Tom M. Apostol, *Introduction to Analytic Number Theory*, 1976, (Springer-Verlag, New York)

[AS64] Milton Abramowitz and Irene A. Stegun, eds., *Handbook of Mathematical Functions*, Dover Publications, New York (1972).

[Ba78a] K. I. Babenko, On a Problem of Gauss, *Soviet Mathematical Doklady* **19**:136-140 (1978)

[Ba78b] K. I. Babenko and S. P. Jur'ev, On the Discretization of a Problem of Gauss, *Soviet Mathematical Doklady* **19**:731-735 (1978)

[Br03] Keith Briggs, *A precise computation of the Gauss-Kuzmin-Wirsing constant*, (2003) (http://keithbriggs.info/documents/wirsing.pdf, unpublished).

[Ed74] H. M. Edwards, *Riemann's Zeta Function*, 1972, (Dover Publications, New York).

[GMP] Free Software Foundation, GNU Multiple Precision Arithmetic Library, version 4.1.4 (2005) http://www.swox.com/gmp/

[Khi35] A. Ya. Khinchin, *Continued Fractions*, (1964) Dover Publications, New York. *(See section 15)*.

[May91] Dieter H. Mayer, Continued fractions and related transformations; appearing in *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*. T. Bedford, M. Keane, C. Series (eds). Oxford University Press, Oxford, 1991.

[Wir74] Eduard Wirising, On the theorem of Gauss-Kusmin-Levy and a Frobenius-type theorem for function spaces, *Acta Arithmetica* **26**:507-528 (1974)

[Kuz28] Kuzmin, Atti del Congresso (1928)

[Rue94] Ruelle, Dynamical Zeta CRMmonographs (1994)