On the Vepstas Representation for the Riemann Zeta Function

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Abstract: We show that the Vepstas expression for the Riemann zeta function implies interesting relations between Stirling, Harmonic and Bernoulli numbers and the Stieltjes constants.

Key words: Riemann zeta function - Stirling numbers - Stieltjes constants - Bernoulli numbers

INTRODUCTION

Vepstas [1] obtained the following representation for the Riemann zeta function [2]:

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \sum_{k=0}^{\infty} (-1)^k \left(\frac{s-1}{k}\right) a_k, \ s \neq 1, \quad (1)$$

which gives the value $\zeta(0) = -\frac{1}{2}$. On the other hand, we have the expansion [2]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (s-1)^r, \ s \neq 1, \quad (2)$$

where the $\gamma_r$ are the Stieltjes constants; from (2) for $s = 0$:

$$\sum_{r=0}^{\infty} \frac{\gamma_{r+1}}{(2r+1)!} = \frac{1}{2} - \sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!}. \quad (3)$$

Now (2) for $s = 2$ implies the relation:

$$\sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!} = \zeta(2) - 1 + \sum_{r=0}^{\infty} \frac{\gamma_{2r+1}}{(2r+1)!},$$

where we can apply (3) to deduce the properties:

$$\sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!} = \frac{1}{2} \zeta(2) - \frac{1}{4} = \frac{\pi^2}{12} - \frac{1}{4} \sum_{r=0}^{\infty} \frac{\gamma_{2r+1}}{(2r+1)!} = \frac{3}{4} - \frac{1}{2} \zeta(2) = \frac{3}{4} - \frac{\pi^2}{12}. \quad (4)$$

We know the following connection between a binomial coefficient and the Stirling numbers of the first kind [2-5]:

$$\binom{s-1}{k} = \frac{1}{k!} \sum_{r=0}^{k} S_k^{(r)} (s-1)^r, \quad (5)$$

Then (1) acquires the structure:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \sum_{r=0}^{\infty} \frac{q_r (s-1)^r}{r!}, \quad (6)$$

Such that:

$$q_r \equiv \sum_{k=r}^{\infty} \frac{(-1)^k}{k!} a_k S_k^{(r)} \quad : \quad q_0 = a_0. \quad (7)$$
The comparison of (2) and (6) gives the value:

\[ \gamma_0 = \frac{1}{2} - q_0 \quad \therefore \quad a_0 = \frac{1}{2} - \gamma_0, \quad (8) \]

where \( \gamma_0 = 0.577215664901 \ldots \) is the famous Euler-Mascheroni’s constant \([2, 6-9]\), thus from (1) for \( s = 2, 3, 4, \ldots [1]\):

\[ a_1 = \frac{1}{2} \zeta(2) - \gamma_0 - \frac{1}{4}, a_2 = \zeta(2) - \frac{1}{3} \zeta(3) + 2 - \gamma_0, a_3 = \frac{3}{2} \zeta(2) - \zeta(3) + \frac{1}{4} \zeta(4) - \gamma_0 - \frac{23}{24}, \ldots \quad (9) \]

with the recurrence relation:

\[ q_{n+1} + q_n = \frac{(-1)^n}{(n+1)!} \gamma_{n+1}, \quad n \geq 0, \quad (10) \]

That is:

\[ q_0 = -\gamma_0 + \frac{1}{2}, q_1 = \gamma_1 + \gamma_0 - \frac{1}{2}, q_2 = -\frac{1}{2} \gamma_2 - \gamma_1 + \gamma_0 + \frac{1}{4}, \ldots \quad (11) \]

The general solution of (10) is given by:

\[ q_n = (-1)^n \left[ \frac{1}{2} - \sum_{r=0}^{\infty} \frac{\gamma_r}{r!} \right], \quad n \geq 0. \quad (12) \]

It is immediate the inversion of (7):

\[ a_r = (-1)^r r! \sum_{k=r}^{\infty} q_k S_k^{[r]}, \quad r \geq 0, \quad (13) \]

Involving Stirling numbers of the second kind \([2-5]\). We know \([2, 3]\) that \( S_k^{[1]} = 1, k \geq 1 \), then from (13):

\[ a_1 = -\sum_{k=1}^{\infty} q_k = -[(q_2 + q_4) + (q_4 + q_8) + (q_6 + q_{10}) + \cdots], \quad (10) \]

\[ = \frac{\gamma_2}{2!} + \frac{\gamma_4}{4!} + \frac{\gamma_6}{6!} + \cdots = \sum_{r=0}^{\infty} \frac{\gamma_r}{(2r)!} - \gamma_0 = \frac{1}{2} \zeta(2) - \gamma_0 - \frac{1}{4}, \quad (4) \]

In agreement with (9).

Vepstas \([1]\) deduced the values:

\[ \sum_{r=0}^{\infty} a_r = \ln(\sqrt{2\pi}) - 1, \sum_{r=0}^{\infty} \frac{\gamma_r}{2^r} = 2 - 3 \ln 2, \quad (14) \]

Thus (13) and (14) imply the relations:

\[ \sum_{k=0}^{\infty} (-1)^k q_k = \ln(\sqrt{2\pi}) - 1 = -0.0810 6146 6795 \ldots, \quad (15) \]

\[ \sum_{k=0}^{\infty} \frac{1 - 2^{k+1}}{k+1} q_k B_{k+1} = 1 - 3 \ln \sqrt{2} = -0.0397 2077 084 \ldots, \quad (16) \]

Involving Bernoulli numbers \([2-5]\); in the deduction of (15) were applied the identities \([2, 3, 10]\):

\[ \sum_{r=0}^{\infty} (-1)^r r! S_k^{[r]} = (-1)^k \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r} r! S_k^{[r]} = \frac{2}{k+1} B_{k+1}. \quad (16) \]

Now we use (7) for \( r = 1, 2 \) and (11) to obtain the expressions:
\[ \sum_{k=1}^{\infty} \frac{a_k}{k} = \frac{1}{2} - \gamma_0 - \gamma_1, \quad \sum_{k=2}^{\infty} \frac{a_k}{k} H_{k-1} = \frac{1}{2} - \gamma_0 - \gamma_1 - \frac{1}{2} \gamma_2, \]  

Involving Harmonic numbers, where we employ the values:

\[ S^{(1)}_k = (-1)^{k-1} (k-1)!, \quad S^{(2)}_k = (-1)^k (k-1)! H_{k-1} . \]  

REFERENCES